

1. **Intuition.** Discuss intuition about finite sets.
2. **Definition.** For  $n \in \mathbb{Z}^+$ , define  $\bar{n}$  to be the set of positive integers which are less than or equal to  $n$ :

$$\bar{n} = \{m \in \mathbb{Z}^+ \mid m \leq n\} = \{1, \dots, n\}$$

Also, define  $\bar{0} = \emptyset$ .

3. **Definition.** A set  $A$  is said to be *finite* if either  $A = \emptyset$  or there exists a bijection  $A \rightarrow \bar{n}$  for some  $n \in \mathbb{Z}^+$ . In the former (resp. latter) case we say that the *cardinality of  $A$  is equal to 0* (resp.  $n$ ), and write  $|A| = 0$  (resp.  $|A| = n$ ).
4. **Lemma.** For  $n \in \mathbb{Z}^+$  and  $j \in \bar{n}$ , there is a bijection

$$\bar{n} - \{j\} \rightarrow \overline{(n-1)}$$

5. **Theorem. (The pigeonhole principle)** Suppose  $m, n \in \mathbb{Z}^+$ . If there is an injection  $\bar{m} \rightarrow \bar{n}$ , then  $m \leq n$ .
6. **Corollary.** The cardinality of a finite set is well-defined. If  $|A| = m$  and  $|A| = n$ , then  $m = n$ .
7. **Theorem.** If  $n \in \mathbb{Z}^+$  and  $A \subset \bar{n}$ , then there exists a bijection  $A \rightarrow \bar{m}$  for some  $m \in \mathbb{N}$  such that  $m \leq n$ .
8. **Corollary.** If  $A \subset B$  and  $B$  is finite, then  $A$  is finite. Furthermore, if  $A$  is a proper subset of  $B$ , then  $|A| < |B|$ .
9. **Lemma.** If  $A$  and  $B$  are disjoint finite sets, then  $A \cup B$  is finite.
10. **Definition.** A set is said to be *infinite* if it is not finite.
11. **Example.** The set  $\mathbb{Z}^+$  is infinite. *There's always room at the Hilbert hotel.*
12. **Theorem.** For a set  $A$ , the following are equivalent.
  - (a)  $A$  is an infinite set.
  - (b) There is an injection  $\mathbb{Z}^+ \rightarrow A$ .
  - (c) There is a bijection  $A \rightarrow S$  where  $S$  is a proper subset of  $A$ .
  - (d) There is a surjection  $A \rightarrow \mathbb{Z}^+$ .

13. **Examples.** More examples of infinite sets.

14. **Intuition.** Discuss intuition about comparing infinite sets. *Two Irish farmers.*

15. **Definition.** Two sets  $A$  and  $B$  have the *same cardinality*, denoted by  $|A| = |B|$ , if there exists a bijection  $f : A \rightarrow B$ . This is the *one cow, one sheep* definition of cardinality.

The cardinality of a set  $A$  is *less than or equal to* the cardinality of a set  $B$ , denoted by  $|A| \leq |B|$ , if there exists an injection  $f : A \rightarrow B$ .

16. **Cardinal Arithmetic.** Prove the following results for **finite** sets. These can be used for definitions of arithmetic of infinite cardinals.

(a) *Sums.* If  $A_1, \dots, A_n$  are disjoint sets, then

$$|A_1 \cup \dots \cup A_n| = |A_1| + \dots + |A_n|$$

(b) *Products.*  $|A \times B| = |A||B|$

(c) *Exponentiation.*  $|B^A| = |B|^{|A|}$ . Recall that the set  $B^A$  is the set of all functions from  $A$  to  $B$ .

(d) *Exponentiation (powers of 2).*  $|\mathcal{P}(A)| = 2^{|A|}$ , where  $\mathcal{P}(A)$  denotes the set of all subsets of  $A$ .

17. **Definition.** A set is said to be *countably infinite* if it has the same cardinality as the positive integers  $\mathbb{Z}^+$ .

A set is said to be *countable* if it is finite, or countably infinite.

18. **Examples.** The following sets are all countably infinite; the set of even integers, the set of odd integers, the set of all integers,  $\mathbb{Z}^+ \times \mathbb{Z}^+$ ,  $\mathbb{Z} \times \mathbb{Z}$ ,  $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ , an  $n$ -fold cartesian product  $\mathbb{Z} \times \dots \times \mathbb{Z}$ ,  $\mathbb{Q}^+$ ,  $\mathbb{Q}$ .

19. **Examples.** Some results.

(a) Show that if  $A$  and  $B$  are disjoint, countable sets, then  $A \cup B$  is also countable.

(b) Show that the same result holds if  $A$  and  $B$  are not necessarily disjoint.

(c) Show that if  $A_1, \dots, A_n$  are  $n$  countable sets, then  $A_1 \cup \dots \cup A_n$  is also countable.

(d) Show that if  $\{A_n\}_{n \in \mathbb{Z}^+}$  is a countable collection of countable sets, then the union

$$\bigcup_{n=1}^{\infty} A_n$$

is countable.

20. **Theorem.** If  $|A| \leq |B|$ , and  $B$  is countable, then  $A$  is also countable.

21. **Examples.** The following sets of real numbers all have the same cardinality;  $\mathbb{R}$ ,  $(0, \infty)$ ,  $(-\pi/2, \pi/2)$ ,  $(0, \pi)$ ,  $(0, 1)$ ,  $(a, b)$  for any  $a < b$ ,  $[a, b)$ ,  $(a, b]$ ,  $[a, b]$ .

22. **Theorem (baby Cantor).** The set  $(0, 1)$  is not countable. In fact, there is no surjective map  $\mathbb{Z}^+ \rightarrow (0, 1)$ .

23. **Definition.** An infinite set which is not countable is said to be *uncountably infinite*. The set  $\mathbb{R}$  is uncountably infinite.

24. **Examples.** Show that the set of all irrational numbers is uncountably infinite. [Hint: points 18, 19, 21 and 22 all combine to give an argument by contradiction]

25. **Definition.** A real number  $r$  is called *algebraic* if it is the root of a polynomial with integer coefficients.

26. **Example.** Show that  $\sqrt{2}$  is an algebraic number. Show that every rational number is algebraic. Show that the set of all algebraic numbers is countably infinite. Therefore, its complement (the set of *transcendental numbers*) is uncountably infinite.

27. **Question.** So “most” real numbers are transcendental. Give some explicit examples of transcendental numbers. Can you give a proof that one of your examples is indeed transcendental?

28. **Theorem (grown up Cantor).** For any set  $A$  we have  $|A| \leq |\mathcal{P}(A)|$  but  $|A| \neq |\mathcal{P}(A)|$ .

29. **Corollary.** There is an infinite hierarchy of cardinalities of infinite sets.

$$|\mathbb{Z}^+| < |\mathcal{P}(\mathbb{Z}^+)| < |\mathcal{P}(\mathcal{P}(\mathbb{Z}^+))| < \dots$$

30. **Theorem (Schröder-Bernstein).** Let  $A$  and  $B$  be sets. If

$$|A| \leq |B| \quad \text{and} \quad |B| \leq |A|$$

then  $|A| = |B|$ .

31. **Example.** Trace through the proof of Schröder-Bernstein above in the case that  $A = B = \mathbb{Z}^+$  and the two injective maps are  $n \mapsto 2n$  and  $n \mapsto 3n$ .

32. **Theorem.** There is an injective map  $\mathcal{P}(\mathbb{Z}^+) \rightarrow (0, 1)$ .

**Theorem.** There is an injective map  $(0, 1) \rightarrow \mathcal{P}(\mathbb{Z}^+)$ .

33. **Corollary.**  $\mathcal{P}(\mathbb{Z}^+)$  has the same cardinality as  $\mathbb{R}$ .

This cardinality is called *the power of the continuum*. It is an axiom of the foundations of mathematics (which is known to be independent of the remaining axioms), that there are no sets with cardinality strictly between  $\mathbb{Z}^+$  and  $\mathbb{R}$ . This axiom is known as *the continuum hypothesis*.

34. **Exercise.** Show that  $\mathbb{R}^2$  and  $\mathbb{R}$  have the same cardinality. [Hint: We already know that there are many injective maps  $\mathbb{R} \rightarrow \mathbb{R}^2$ . If only we could find an injective map going the other direction, we'd be done by Schröder-Bernstein...]

35. **Remark.** There are pretty examples of surjective maps  $\mathbb{R} \rightarrow \mathbb{R}^2$  due to Peano and to Hilbert. Do a Google search on Peano curve and on Hilbert curve. What has this got to do with the OU MathClub blog?

36. **Weird scenes inside the goldmine...[The Cantor Set].** Let  $A_0 = [0, 1]$ , and for each  $n \geq 1$  define

$$A_n = A_{n-1} \setminus \bigcup_{k=0}^{\infty} \left( \frac{1+3k}{3^n}, \frac{2+3k}{3^n} \right)$$

and finally define the Cantor set,  $C$ , to be the intersection

$$C = \bigcap_{n \in \mathbb{Z}^+} A_n$$

(a) Show that  $C$  is uncountable.

(b) Show that  $C$  has *no length!*. [Hint:  $C$  is a subset of each  $A_n$ . Show that the total length of a given  $A_n$  is  $(2/3)^n$ ]

37. **Example.** Show that there exists uncountably many disjoint circles in the plane. What about figure eights?

**Remarks.** The *generalized continuum hypothesis* states that there are no sets with cardinality strictly between any of the sets listed in Corollary 29 above. It is independent of the axioms of set theory.

The result of Exercise 34 is remarkable. It is hard to imagine a bijective map from  $\mathbb{R}$  to  $\mathbb{R}^2$ , even a *surjective* map seems unreasonable. There is a beautiful map due to Peano (called the Peano curve) which is a *continuous map* with domain the interval  $[0, 1]$  and range the 2-dimensional square  $[0, 1]^2$ . If this sounds *way out there* but still strangely compelling, you might want to consider taking a *Topology* or an *Analysis* course. In these courses, you would also learn lots more about the Cantor set from Example 36.