Q1) [20 points] Define what it means for a function \( f : A \rightarrow B \) to be surjective.

\[
f : A \rightarrow B \text{ is surjective if } \forall b \in B \exists a \in A \text{ such that } f(a) = b.
\]

Recall that \((0,1)\) denotes the interval \( \{x \in \mathbb{R} \mid 0 < x < 1\} \), and that \( \mathbb{Z}^+ \) denotes the set of positive integers. Give a detailed proof that no function

\[ f : \mathbb{Z}^+ \rightarrow (0,1) \]

can be surjective. (This is the usual Cantor diagonalization argument that \((0,1)\) is uncountable).

Every \( x \in (0,1) \) admits a decimal expansion which does not end in an infinite string of 9's. Work with these expansions throughout this proof.

Given \( f : \mathbb{Z}^+ \rightarrow (0,1) \) write out the unique decimal expansions for \( f(n) \) for all \( n \in \mathbb{Z}^+ \):

\[
\begin{align*}
f(1) &= 0. a_{11} a_{12} a_{13} \ldots \\
f(2) &= 0. a_{21} a_{22} a_{23} \ldots \\
f(3) &= 0. a_{31} a_{32} a_{33} \ldots \\
&\vdots \\
f(n) &= 0. a_{n1} a_{n2} a_{n3} \ldots \\
&\vdots
\end{align*}
\]

define a number \( b = 0.b_1 b_2 b_3 \ldots \) so that

\[
\begin{align*}
&b_i \neq a_{ii} \quad \forall i \\
&b_i \neq 9 \quad \forall i \\
&b_i \neq 0 \quad \forall i
\end{align*}
\]

Clearly \( 0.b_1 b_2 \ldots \in (0,1) \) — we've ruled out 0.000\ldots & 0.999\ldots & by construction \( 0.b_1 b_2 \ldots \neq f(\mathbb{Z}^+) \)

\[ \Rightarrow f \text{ not surjective.} \]
Q2]...[20 points] Prove that there is a bijection between the set $\mathbb{Z}$ and the set $\mathbb{Z}^+$.

$$f: \mathbb{Z} \rightarrow \mathbb{Z}^+ : n \mapsto \begin{cases} 2n+1 & \text{if } n \geq 0 \\ -2n & \text{if } n < 0 \end{cases}$$

is a bijection.

$f$ is injective: Given $f(x) = f(y)$.

- **Case 1:** odd output $\Rightarrow 2x + 1 = 2y + 1$ \Rightarrow $2x = 2y$ \Rightarrow $x = y$.
- **Case 2:** even output $\Rightarrow -2x = -2y$ \Rightarrow $x = y$.

In other case, $x = y$. \Rightarrow $f$ injective.

$f$ is surjective: Given $m \in \mathbb{Z}^+$.

- **Case 1:** $M$ is even $\Rightarrow M = 2k$ some $k \in \mathbb{Z}^+$ $\Rightarrow M = 2(-k) = f(-k)$.
- **Case 2:** $M$ is odd $\Rightarrow M = 2l + 1$ some $l \in \mathbb{Z}$ $\Rightarrow M = f(l)$.

Prove that there is a bijection between the set $\mathbb{Z}^+ \times \mathbb{Z}^+$ and the set $\mathbb{Z}^+$.

**Method 1**

$$\begin{array}{ccc}
(1,1) & (1,2) & (1,3) \\
(2,1) & (2,2) \\
(3,1) & & \\
\end{array}$$

indicates a bijection $\mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$.

But it takes a bit of care to prove that one exists from this perspective.

**Step 1** Let $A_n = \{(p, q) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid p + q = n + 1\}$

$A_1 = \{ (0, 1) \}$, $A_2 = \{ (1, 2), (2, 1) \}$, $A_3 = \{ (1, 3), (3, 1), (2, 2) \}$

Note $|A_n| = n$ & $A_n$ are disjoint sets.

$\& \bigcup_{n=1}^{\infty} A_n = \mathbb{Z}^+ \times \mathbb{Z}^+$.

**Step 2** Let $B_n = \{ \frac{n(n+1)}{2} + 1, \ldots, \frac{(n+1)(n+2)}{2} \} \in \mathbb{Z}^+$

$B_1 = \{ 1 \}$, $B_2 = \{ 2, 3 \}$, $B_3 = \{ 4, 5, 6 \}$, $B_4 = \{ 7, 8, 9, 10 \}$

Note $|B_n| = n$, the $B_n$ are disjoint and $\bigcup_{n=1}^{\infty} B_n = \mathbb{Z}^+$. 

Note: $\frac{n(n+1)}{2} + \frac{(n+1)(n+2)}{2} = \frac{(n+1)n(n+2)}{2}$. 

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Since \(|A_n| = n = |B_n| \) \(\Rightarrow\) there exist bijections

\[ A_n \xrightarrow{f_n} \overline{n} \] \[ B_n \xrightarrow{g_n} \overline{n} \]

\(\Rightarrow\) \(g_n \circ f_n : A_n \rightarrow B_n\) is a bijection.

Now define \(F : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+\) by

\[ F \mid_{A_n} = g_n \circ f_n \]

Check \(F\) is a bijection.

<table>
<thead>
<tr>
<th>F surjective</th>
<th>Given (m \in \mathbb{Z}^+)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Rightarrow) (m \in B_n) for some (n)</td>
<td></td>
</tr>
<tr>
<td>(\Rightarrow) (M = g_n \circ f_n(a)) for some (a \in A_n \leq \mathbb{Z}^+ \times \mathbb{Z}^+)</td>
<td></td>
</tr>
<tr>
<td>(\Rightarrow) (M = F(a)) for some (a \in \mathbb{Z}^+ \times \mathbb{Z}^+)</td>
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<table>
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<tr>
<th>F injective</th>
<th>(F(a) = F(b))</th>
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</thead>
<tbody>
<tr>
<td>(\Rightarrow) (F(a) = F(b)) lie in some (B_n).</td>
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</tr>
<tr>
<td>(\Rightarrow) (a, b) must lie in (A_n)</td>
<td></td>
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<tr>
<td>(&amp;) (F(a) = F(b))</td>
<td></td>
</tr>
<tr>
<td>(\Rightarrow) (g_n \circ f_n(a) = g_n \circ f_n(b))</td>
<td></td>
</tr>
<tr>
<td>(\Rightarrow) (a = b) since (g_n \circ f_n) bij.</td>
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</tbody>
</table>

Method (2)

Now apply Schröder-Bernstein!

\(f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \times \mathbb{Z}^+ : n \mapsto (n, 1)\) is clearly an injection.

\(g : \mathbb{Z}^+ \times \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ : (m, n) \mapsto 2^m 3^n\) is also an injection.

\[ g(m, n) = g(a, b) \Rightarrow 2^m 3^n = 2^a 3^b \]

\(m = a \Rightarrow \) divide both sides by \(2^m\) to get \(3^a = 3^b\)

\(\Rightarrow\) \(n = b\) (take logs (\(\cdot\))).

\(\Rightarrow\) \((m, n) = (a, b)\)

\(\Rightarrow\) \(g\) injective.

\(\uparrow\) If \(m = a\) then \(m < a\) (without loss of generality)

\(\Rightarrow \) divide both sides by \(2^m\) to get \(3^a = 2^{a-m} 3^b\)

\(\Rightarrow\) odd \(\uparrow\) even \(\Rightarrow\) contradiction

\(\Rightarrow\) \(m = a\).
Q3]... [24 points] Define what it means for the set \( A \) to be countable.

\[
A \text{ is countable if } A \text{ is finite or } \exists \text{ bijection } \mathbb{Z}^+ \rightarrow A.
\]

Define what it means for the set \( A \) to be uncountable.

\[
A \text{ is uncountable if } A \text{ is not countable}. \text{ This means, } A \text{ is not finite and } \nexists \text{ bijection } \mathbb{Z}^+ \rightarrow A.
\]

Say whether each of the following sets is countable or not.

1. The set \( \mathbb{Z} \) of all integers.
   \[
   \mathbb{C} \quad \text{(see Q2!)}
   \]

2. The set \( \mathbb{R} \) of all real numbers.
   \[
   \mathbb{U} \quad \text{(see Q1!)}
   \]

3. The set \( \mathbb{Q} \) of all rational numbers.
   \[
   \mathbb{C} \quad \text{(see Q2!)}
   \]

4. The set of all irrational numbers.
   \[
   \mathbb{U} \quad \text{(since } \{\text{irrational}\} \cup \mathbb{Q} = \mathbb{R})
   \]

5. The set of all functions from \( \{1\} \) to \( \mathbb{R} \).
   \[
   \leftrightarrow_{\mathbb{R}} \quad \mathbb{R} \rightarrow \mathbb{U}
   \]

6. The set of all functions from \( \mathbb{R} \) to \( \{1\} \).
   \[
   \leftrightarrow_{\mathbb{R}} \quad \mathbb{I} \rightarrow \mathbb{C}
   \]

7. The set of all functions from \( \{1, 2, 3\} \) to \( \mathbb{Z} \).
   \[
   \leftrightarrow_{\mathbb{bij}} \quad \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \Rightarrow \text{countable}, \mathbb{C}
   \]

8. The set of all functions from \( \mathbb{Z} \) to \( \{1, 2, 3\} \).
   \[
   \mathbb{U} \quad \text{do diagonal argument}
   \]

9. The set of all lines in the plane \( \mathbb{R}^2 \).
   \[
   \leftrightarrow_{\mathbb{bij}} \quad \mathbb{R} \times \mathbb{R} \Rightarrow \mathbb{U}
   \]

10. The power set \( \mathbb{P}(\mathbb{Z}) \) of \( \mathbb{Z} \).
    \[
    \mathbb{U}
    \]
Q4]... [36 points] True or False.

1. The composition of reflections in two intersecting lines is a rotation.  \( T \)

2. The set of symmetries of a regular pentagon (5 sides) has 10 elements.  \( T \)  5 reflections 5 rotations

3. The set of symmetries of a regular polygon with 1,000 sides is countable.  \( T \)  has 2,000 elements!

4. The set of symmetries of a circle is countable.  \( F \)  (has rotations through \( \theta \) \( \forall \theta \in [0,2\pi) \))

5. \( \text{Perm} \{1,2,\ldots,n\} \) has \( n^n \) elements.  \( F \)  has \( n! \) elements

6. \( \text{Perm} (\mathbb{Z}^+) \) is countable.  \( F \)  (see below (*) )

7. \((123)(245)(132) = (345).  \( T \)  (just do it!)

8. If \( m \) is reflection in some line, and \( R \) is a 90° counterclockwise rotation about a point \( O \), then \( mRm \) is a 90° counterclockwise rotation about the point \( m(O) \).  \( F \)  It is a clockwise rotation about \( m(O) \).

9. \((12)(23)(34)(45)(56)(67)(78) = (12345678) \)  \( T \)  (just do it!)

10. \((12)(23)(34)(45)(34)(23)(12) = (15).  \( T \)  (just do it!)

11. The composition of reflections in the three sides of a triangle (taken in any order) is a rotation.  \( F \)  (odd \# of reflections in lines can never yield a rotation!)

12. The composition of reflections in the four sides of a rectangle (taken in any order) is a translation.  \( T \)

(*) \( \text{let } \sigma_1 = (12), \quad \sigma_2 = (34) \ldots, \quad \sigma_n = (2n-1,2n) \ldots \in \text{Perm}(\mathbb{Z}^+)

\text{Then } \text{Perm}(\mathbb{Z}^+) \text{ contains all } \infty \text{ strings } j \text{ the form } \sigma_1^{a_1} \sigma_2^{a_2} \sigma_3^{a_3} \ldots \text{ where } a_i \in \{0,1,2\}

\& \text{ we can do a diagonal argument to show this set is uncountable}.!!