Q1] [12 points] Find a disjunctive normal form expression (involving $\land$, $\lor$, $\neg$, and $P$, $Q$, $R$) which has the following truth table. Show the steps of your work.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$R$</th>
<th>$P\land Q\land R$</th>
<th>$P\land Q\land \neg R$</th>
<th>$\neg P\land Q\land R$</th>
<th>$\neg P\land Q\land \neg R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
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</tbody>
</table>

Step 1: Find expressions whose truth tables give a $T$ in the appropriate row & $F$'s elsewhere. There are 4 expressions for this example.

Step 2: Take the disjunction of the expressions in step 1.

\[
\text{dnf} = (P\land Q\land R) \lor (P\land Q\land \neg R) \lor (\neg P\land Q\land R) \lor (\neg P\land Q\land \neg R)
\]

Find a conjunctive normal form expression (involving $\land$, $\lor$, $\neg$, and $P$, $Q$, $R$) which has the same truth table above. Show the steps of your work.

Step 1: Negate the output column

\[
\begin{array}{c|c|c|c}
  & F & F & T \\
\hline
P & T & T & T \\
Q & T & T & T \\
R & T & T & T \\
\hline
\end{array}
\]

Step 2: Write dnf for the new output column:

\[
(P\land Q\land \neg R) \lor (P\land Q\land \neg R) \lor (\neg P\land Q\land R) \lor (\neg P\land Q\land \neg R)
\]

Step 3: Negate this dnf ... use deMorgan ... get cnf for original table!

\[
(\neg P \lor \neg Q \lor R) \land (\neg P \lor Q \lor \neg R) \land (P \lor Q \lor \neg R) \land (P \lor Q \lor R)
\]
Q2]...[11 points] Write down the distributive law for \( \land \) over \( \lor \).

\[
A \land (B \lor C) \equiv (A \land B) \lor (A \land C)
\]

Write down the distributive law for \( \lor \) over \( \land \).

\[
A \lor (B \land C) \equiv (A \lor B) \land (A \lor C)
\]

Write down the two De Morgan laws (involving negations of \( \land \) and \( \lor \) statements).

\[
\neg(A \land B) \equiv \neg A \lor \neg B
\]

\[
\neg(A \lor B) \equiv \neg A \land \neg B
\]

Use the De Morgan and distributive laws to show that the expression

\[
[P \land (\neg Q) \land R] \lor [P \land (\neg Q) \land (\neg R)] \lor [P \land Q \land R] \lor \neg [(\neg P) \lor (\neg Q) \lor R]
\]

is logically equivalent to \( P \).

Expression \( \equiv \)

\[
(P \land \neg Q \land R) \lor (P \land \neg Q \land \neg R) \lor (P \land Q \land R) \lor (P \land Q \land \neg R)
\]

\(\equiv\) \( P \land \left[ (\neg Q \land R) \lor (\neg Q \land \neg R) \right] \lor (Q \land R) \lor (Q \land \neg R) \)

\(\equiv\) \( P \land \left[ (\neg Q \land (R \lor \neg R)) \lor (Q \land (R \lor \neg R)) \right] \)

\(\equiv\) \( P \land \left[ (\neg Q \land \top) \lor (Q \land \top) \right] \)

\(\equiv\) \( P \land (\neg Q \lor Q) \)

\(\equiv\) \( P \land \top \)

\(\equiv\) \( P \quad \text{done!} \)
Q3]...[12 points] Are the following two expressions logically equivalent. If you say so, please explain why. If you say not, then please give an example which shows that they are different.

\[ \forall x [P(x) \rightarrow Q(x)] \]

and

\[ (\forall x P(x)) \rightarrow (\forall x Q(x)) \]

**No**  
Example (in class!)  
**Universe** = all integers  
\[ P(x) = x \text{ is even} \]  
\[ Q(x) = x \text{ is odd} \]

Statement 1 is **false** : e.g. \( 2 \) is even, but not odd.  
Statement 2 is automatically **true**, since the hypothesis "every integer is even" is false.

Same question for the expressions

\[ \exists x [P(x) \lor Q(x)] \]

and

\[ (\exists x P(x)) \lor (\exists x Q(x)) \]

**Yes**

1st \[ \rightarrow \] 2nd

\[ \exists x \text{ such that } P(x) \lor Q(x) \text{ true} \]

implies \( P(x) \text{ true } \) or \( Q(x) \text{ true for some value of } x \)

\[ \Rightarrow \exists x P(x) \text{ or } \exists x Q(x) \]

2nd \[ \rightarrow \] 1st

\[ \exists x P(x) \lor \exists x Q(x) \]

\[ \Rightarrow \exists x P(x) \]

\[ \Rightarrow P(x) \text{ true} \]

\[ \Rightarrow \exists x P(x) \lor Q(x) \text{ true} \]

\[ \Rightarrow \exists x (P(x) \lor Q(x)) \]
Q4]...[15 points] Give a direct proof of the following. If \( m \) and \( n \) are odd integers, then their product is also odd.

\[
\begin{align*}
m \text{ odd } & \Rightarrow m = 2k+1 \quad \text{for some integer } k. \\
n \text{ odd } & \Rightarrow n = 2l+1 \quad \text{for some integer } l.
\end{align*}
\]

\[
\Rightarrow mn = (2k+1)(2l+1) = 4kl + 2k + 2l + 1
\]

\[= 2(2kl + k + l) + 1\]

which is of the form \( 2(\text{integer}) + 1 \)

\[\Rightarrow \text{ is odd .} \]

Write down the contrapositive of the following statement about integers \( n \). If \( n^3 \) is even, then \( n \) is also even.

If \( n \) is odd, then \( n^3 \) is odd .

Prove the statement “If \( n^3 \) is even, then \( n \) is also even” by giving a proof of its contrapositive.

Start with \( n \) is odd

\[
\Rightarrow n^2 = n \cdot n = \text{product of 2 odd integers is odd (by 1st part above)}
\]

\[
\Rightarrow n^3 = n^2 \cdot n = \text{product of 2 odd integers is odd (by 1st part above)}
\]

\[\Rightarrow n^3 \text{ odd .} \]
Q5]...[15 points] Give a proof of the following: The cube root of 2 is irrational. You are free to cite the results of Q4 if they are of any help to you.

Proof by contradiction.

Assume \( \sqrt[3]{2} \) is rational.

Thus \( 3\sqrt[3]{2} = \frac{p}{q} \) for \( p, q \in \mathbb{Z}^+ \).

By dividing numerator + denom by some power of 2, we may assume that at least one of \( p, q \) is odd.

\[
2 = \frac{p^3}{q^3}
\]

\[2q^3 = p^3 \]

LHS is even \( \Rightarrow \) \( p^3 \) is even
\[ \Rightarrow p \text{ is even} \quad \text{by Q4}. \]

Writing \( p = 2k \) for some integer \( k \), we get

\[2q^3 = (2k)^3 = 8k^3\]
\[\Rightarrow q^3 = 4k^3\]

RHS is even \( \Rightarrow q^3 \) is even
\[\Rightarrow q \text{ is even} \quad \text{by Q4}. \]

So we have both \( p \) and \( q \) are even \( \Rightarrow \) contradicts \( \otimes \).

So \( 3\sqrt[3]{2} \) must be irrational. \( \Box \)
Q6] [20 points] State the principle of induction.

\[ P(n) = \text{statement involving } n. \]

- \( P(1) \) true
- \( \forall k \left[ P(k) \implies P(k+1) \right] \implies P(n) \text{ true } \forall n \in \mathbb{Z}^+ \).

Give a proof by induction of the following. For each positive integer \( n \),

\[ P(n) : \quad 1^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6} \]

**Proof**

1. **\( P(n) \) is true**:
   \[ 1^2 = 1 = \frac{1(1+1)(2(1)+1)}{6} = \frac{1(2)(3)}{6} = 1 \]
   \[ 1 = 1 \quad \checkmark \quad \text{true} \]

2. **\( \forall k \left[ P(k) \implies P(k+1) \right] \)**:
   Assume \( P(k) \) true :
   \[ 1^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6} \]

   Then
   \[ 1^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \]

   \[ = \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \]

   \[ = \frac{(k+1) \left( k(2k+1) + 6(k+1) \right)}{6} \]

   \[ = \frac{(k+1) \left( 2k^2 + 7k + 6 \right)}{6} \]

   \[ = \frac{(k+1)(k+2)(2k+3)}{6} \]
\[ \frac{(k+1)(k+1)+1)}{6} \]

\& so \( P(k+1) \) holds.

By the principle of induction, \( P(n) \) true \( \forall n \in \mathbb{Z}^+ \).
Q7. [15 points] Give a proof by induction of the following. \(2^{2n-1} + 3^{2n-1}\) is a multiple of 5 for all integers \(n \geq 1\).

\[ P(n) : \quad 2^{2n-1} + 3^{2n-1} \text{ is a multiple of 5} \]

\[ P(1) \text{ true:} \quad 2^{2(1)-1} + 3^{2(1)-1} = 2^1 + 3^1 = 2 + 3 = 5 \quad \text{is a multiple of 5} \quad \text{...(5/1).} \]

\[ \forall k (P(k) \Rightarrow P(k+1)) \]

Assume \(P(k)\) true.

\[ 2^{2k-1} + 3^{2k-1} = 5M \quad \text{for some integer } M. \]

Now \[ 2^{2(k+1)-1} + 3^{2(k+1)-1} \]

\[ = 2^{2k-1} + 2 + 3^{2k-1} + 2 \]

\[ = 2 \cdot 2^{2k-1} + 3 \cdot 3^{2k-1} \]

\[ = 4 \cdot 2^{2k-1} + 9 \cdot 3^{2k-1} \quad \text{\( \text{\( \Rightarrow \)} \text{\( = 4 + 5 \) \( \text{by \( P(k) \) true.} \)} \text{\( \text{\( \Rightarrow \)} \text{\( \text{\( = (5M) + 5 \cdot 3^{2k-1} \quad \text{\( \Rightarrow \) by \( P(k) \) true.} \)} \text{\( \text{\( \Rightarrow \) \( P(k+1) \) true.} \)} \text{\( \text{\( = \text{multiple of 5.} \) \( \Rightarrow \) \( P(k+1) \) true.} \)} \text{\( \Rightarrow \) \( P(k+1) \) true.} \)

By induction, \( P(n) \) true for all \( n \in \mathbb{Z}^+ \).