Math 2433-001 Extra Homework I Fall 2008

Conic Sections

In the textbook we encountered *conic sections* as curves which are given either by a *focus-locus* description or by a *focus-directrix* description. For example, the circle is the *locus* of points which are a constant distance from a given point or *focus*. Similarly, the ellipse is the *locus* of points such that the sum of their distances form a given pair of points (called *foci*) is constant. The parabola on the other hand is defined as the *locus* of points whose distance from a fixed point (*focus*) is equal to the distance from a fixed line (*directrix*).

In this homework, you will investigate which conic sections admit *focus-locus* descriptions and which admit *focus-directrix* descriptions. You will investigate the connection between these descriptions, by thinking about the curves as *sections* of a right circular *cone* in 3–dimensions, obtained by intersecting the cone with various planes.

Q1. For each of the sections *ellipse* and *hyperbola* in row 1 of the table below, give a geometric proof (eg. using Dandelin spheres) that they have a focus-locus description (row 3) and a focus-directrix description (row 5). For the focus-directrix case, you should observe that the plane which contains the intersection of a Dandelin sphere and the cone and the plane which contains the conic section will meet in a line. This line will be the directrix you need.

Recall that a Dandelin sphere is one which is tangent to the cone (in a circle) and to the plane which contains the conic section. If you are unfamiliar with the geometry of Dandelin spheres, you should Google the phrase to see some interesting sites.

Q2. Give a proof that the *parabola* has a focus-directrix description as in row 5 of the table. Hint: the plane that contains the intersection of a Dandelin sphere with the cone and the plane which contains the parabola meet in a line. Prove that this line is the directrix of the parabola.

Q3. For each of ellipse, parabola and hyperbola, construct a 3–dimensional paper/cardboard model of the cone(s), section planes, and planes containing the intersections of the cone(s) and Dandelin spheres. Your model should have the conic highlighted, and should have the focus (foci) and directrix highlighted. Your model does not need to have Dandelin spheres (it's hard to construct a sphere from paper/cardboard!). However, the relative positioning of the focus, the directrix, and the horizontal circles of intersection of Dandelin spheres with cone, etc should all be **precise** on your model. Add triangles to your model which help exhibit some of the geometric arguments in Q1 and Q2 above.

Q4. Check that the *circle* is a limiting case of the ellipse (as the plane's normal direction becomes parallel to the cone axis). What happens to the directrix during this limiting process?

Q5. Find a formula (in a and b) for the eccentricity ϵ of an ellipse in standard cartesian form (row 6 of the table). Show your work.

Q6. Ripples/waves–I. Draw concentric circles of radii $1, 2, 3, \ldots, 8$ about two foci which are a distance 5 apart. Find intersection points with the property that the sum of distances from the two foci is 6. What curve do these intersection points lie on? Draw it in on your diagram.

Q7. Ripples/waves–II. Draw a directrix D and a focus F on the plane. Draw lines parallel to D at distance $1, 2, 3, \ldots$. Draw concentric circles about F of radii $\epsilon, 2\epsilon, 3\epsilon, \ldots$ for some number ϵ between 0.5 and 1. For each m find the intersection points of a circle of radius $m\epsilon$ with a line of distance m from D. What curve do these intersection points lie on? Draw it in on your diagram.

	Circle	**Ellipse**	**Parabola**	**Hyperbola**
Cone-plane intersection description	Plane's normal is parallel to the cone axis	Plane's normal makes makes a positive angle with cone axis. Angle is less than cone angle.	Plane's normal makes an angle with the cone axis which is equal to cone angle.	Plane's normal makes an angle with cone axis which is greater than the cone angle.
Has a focus-locus description	YES	YES	NO	YES
	$\|PF\| = C$	$\|PF_1\| + \|PF_2\| = C$	—	$\|PF_1\| - \|PF_2\| = C$
Has a focus-directrix description	NO	YES	YES	YES
	—	$\|PF_1\| = \epsilon\|PD\|$ $0 < \epsilon < 1$	$\|PF_1\| = \|PD\|$	$\|PF_1\| = \epsilon\|PD\|$ $\epsilon > 1$
Standard cartesian description	$x^2 + y^2 = a^2$	$(x/a)^2 + (y/b)^2 = 1$ $a \geq b > 0$ or $b \geq a > 0$	$x^2 = \pm 4py$ or $y^2 = \pm 4px$	$(x/a)^2 - (y/b)^2 = 1$ or $(y/b)^2 - (x/a)^2 = 1$
Standard polar description	$r = l/(1 \pm \epsilon \cos(\theta))$ or $r = l/(1 \pm \epsilon \sin(\theta))$ with $\epsilon = 0$	with $0 < \epsilon < 1$	with $\epsilon = 1$	with $\epsilon > 1$