Math 2433-006 Final Examination Name:________________________

Wednesday, May 7, 2003, 8:00am–10:00am.

120 points total. Answer all ten numbered questions for full credit.

Q1]..[10 points] Give spherical and cylindrical coordinates equations describing the following cone in 3-dimensions: cone obtained by rotating the half-line $x = z$, $y = 0$, $x \geq 0$ about the z-axis.

- **cylindrical** [you will need an equation and an inequality]

- **spherical** [you will only need an equation]

Q2].[10 points] Find the equation of the plane which contains the line $x = 3 + 2t$, $y = t - 1$, $z = -t$ and the point $(0, 1, 0)$.

Q3]..[20 points] Give a geometric argument (proof) for why the formula for the distance d from the point $\mathbf{p}$ to the line $\mathbf{r} = \mathbf{r_0} + t\mathbf{v}$ is given by

$$d = \frac{|(\mathbf{p} - \mathbf{r_0}) \times \mathbf{v}|}{|\mathbf{v}|}$$

Use this formula to find the distance from the origin to the line $x = 3 + 2t,\ y = t - 1,\ z = -t$.

Q4]..[10 points] For each of the two questions below, choose one answer from the given list.

The polar curve $r = \sin(6\theta)$ is a "rose" with how many "leaves"?

1. 3 leaves
2. 6 leaves
3. 12 leaves

The polar curve $r = \cos(5\theta)$ is a "rose" with how many "leaves"?

1. 5 leaves
2. 10 leaves
3. uncountably infinitely many leaves

Q5]..[10 points] Sketch (not too much detail) the graph of the cardioid $r = 1 - \sin\theta$ and compute the area contained inside it.

Q6]..[20 points] You are asked to fill in the steps in the argument below to obtain the formula

$$\kappa = \frac{|\dot{\mathbf{r}} \times \ddot{\mathbf{r}}|}{|\dot{\mathbf{r}}|^3}$$

for the curvature κ of a parametric curve $\mathbf{r}(t)$ in 3-dimensions. Recall that $\kappa = |\frac{d\hat{\mathbf{T}}}{ds}|$ where arc-length s is defined by $\frac{ds}{dt} = |\dot{\mathbf{r}}|$.

By definition of the unit tangent vector $\hat{\mathbf{T}}$ we have

$$\dot{\mathbf{r}} = \frac{ds}{dt}\hat{\mathbf{T}}$$

1. Derive the expression

$$\ddot{\mathbf{r}} = \frac{d^2 s}{dt^2}\hat{\mathbf{T}} + \left(\frac{ds}{dt}\right)^2 \frac{d\hat{\mathbf{T}}}{ds}$$

2. Now show that

$$\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \left(\frac{ds}{dt}\right)^3 \hat{\mathbf{T}} \times \frac{d\hat{\mathbf{T}}}{ds}$$

3. Finally, since $|\hat{\mathbf{T}}| = 1$ we know that $\frac{d\hat{\mathbf{T}}}{ds}$ is perpendicular to $\hat{\mathbf{T}}$ (why?), and so we can take lengths of either side of the expression above to get the final answer (show the work).

Q6]..(continued) Use the curvature formula to compute the curvature at any point of the helix

$$\mathbf{r}(t) = \langle a\cos t, a\sin t, bt\rangle$$

Verify that your answer does not depend on the particular point of the helix.

Q7]..[10 points] Determine whether the following series is convergent or divergent, stating clearly what tests (etc) that you used.

$$\sum_{n=1}^{\infty} \frac{tan^{-1}(n)}{n^2+1}$$

Q8]..[10 points] Determine if the following series is absolutely or conditionally convergent.

$$\sum_{n=1}^{\infty} \cos(n\pi) \frac{n^2 2^n}{n!}$$

Q9]..[10 points] By differentiating the geometric series, find a function which is represented by the power series

$$\sum_{n=1}^{\infty} nx^n$$

Use your answer to find the exact value of the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{3^n}$$

Q10]..[10 points] Find the Taylor series for $f(x) = \sin x$ about the point π. Show your work.

— (2003) — Solns —

Q1 Not covered (Now in Calc IV)

Q2
Q3
Q6
Q7
Q8
} Covered on Thursday's class

Q9 Geometric Series $\frac{1}{1-x} = 1 + x + x^2 + \cdots$ $(|x| < 1)$

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = 0 + 1 + 2x + 3x^2 + \cdots \quad (|x|<1)$$

$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1}$$

Times x ⇒

$$\boxed{\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} n x^n} \quad (|x|<1)$$

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{3^n} = -\sum_{n=1}^{\infty} (-1)^n \frac{n}{3^n} = -\sum_{n=1}^{\infty} n\left(-\frac{1}{3}\right)^n$$

Since $\left|-\frac{1}{3}\right| = \frac{1}{3} < 1$ we can use our series ....

$x = -\frac{1}{3}$

$$\frac{\left(-\frac{1}{3}\right)}{\left(1-\left(-\frac{1}{3}\right)\right)^2} = \sum_{n=1}^{\infty} n\left(-\frac{1}{3}\right)^n$$

So our sum $= -\sum_{n=1}^{\infty} n\left(-\frac{1}{3}\right)^n = -\left(\frac{\left(-\frac{1}{3}\right)}{\left(1-\left(-\frac{1}{3}\right)\right)^2}\right)$

$$= \frac{+\frac{1}{3}}{\left(\frac{4}{3}\right)^2}$$

$$= \frac{1}{\not{3}} \cdot \frac{\not{3}}{4} \cdot \frac{3}{4} = \boxed{\frac{3}{16}}$$

Q10

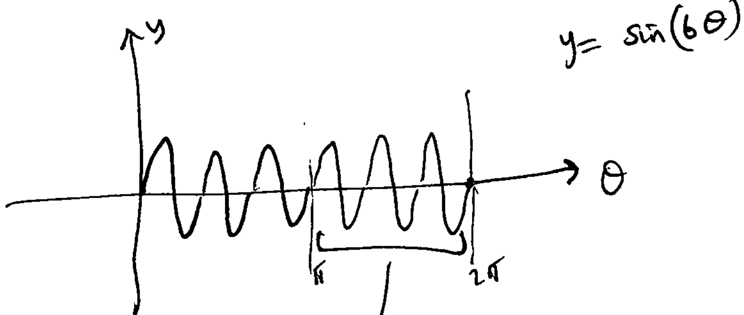
	$x = \pi$
$\dots = f^{(4)}(x) = f(x) = \sin(x)$	0
$\dots = f^{(5)}(x) = f'(x) = \cos(x)$	-1
$\dots = f^{(6)}(x) = f''(x) = -\sin(x)$	0
$\dots = f^{(7)}(x) = f'''(x) = -\cos(x)$	$+1$

$$c_n = \frac{f^{(n)}(\pi)}{n!} = \begin{cases} 0 & n \text{ even} \\ -1/n! & n \text{ has remainder 1 on} \div \text{by 4} \\ +1/n! & n \text{ has remainder 3 on} \div \text{by 4} \end{cases}$$

Series = $0 - (x-\pi) + 0 + \frac{(x-\pi)^3}{3!} + 0 - \frac{(x-\pi)^5}{5!} + \cdots$

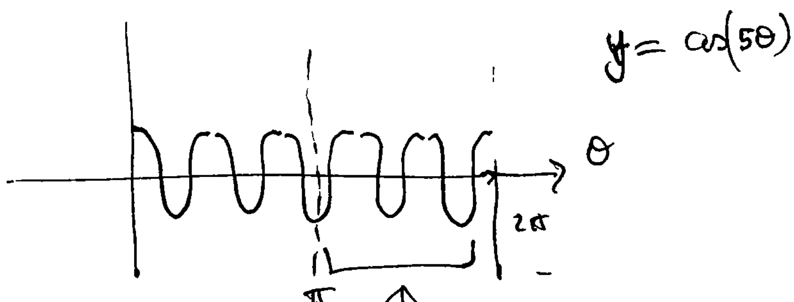
$$\sin(x) = -(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!} + \frac{(x-\pi)^7}{7!} - \cdots$$

Q4



shift by π & reflect in θ-axis ⇒ get 2× as many loops

12 leaves

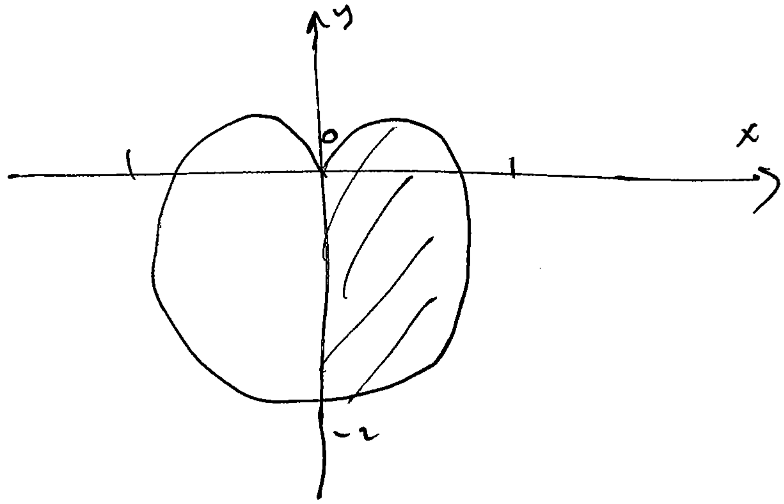


shift by π & flip in θ-axis ⇒ same # of loops → just traversed twice

⇒ **5 leaves**

Q5

$r = 1 - \sin\theta$



$$\text{Area} = 2\left(\text{Area of RHS (shaded)}\right)$$

$$= 2\int_{-\pi/2}^{\pi/2} \left(\text{Area element}\right)$$

$$= \cancel{2}\int_{-\pi/2}^{\pi/2} \frac{r^2}{\cancel{2}}\, d\theta$$

$$\sin^2\theta = \frac{1-\cos(2\theta)}{2}$$

$$= \int_{-\pi/2}^{\pi/2} (1-\sin\theta)^2\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left(1 + \sin^2\theta - 2\sin\theta\right) d\theta = \int_{-\pi/2}^{\pi/2} \left(\frac{3}{2} - \frac{\cos(2\theta)}{2} - 2\sin\theta\right) d\theta$$

$$= \left[\frac{3\theta}{2} - \frac{\sin(2\theta)}{4} + 2\cos\theta \right]_{-\pi/2}^{\pi/2}$$

$$= \frac{3\pi}{2}$$