Q1... [10 points] Differentiate the following function

\[ f(x) = \tan \left( \sqrt{x^2 + 4x + 1} \right) \]

\[
\begin{align*}
f'(x) & = \frac{d}{dx} \left( \tan \left( \sqrt{x^2 + 4x + 1} \right) \right) \\
& = \sec^2 \left( \sqrt{x^2 + 4x + 1} \right) \cdot \frac{d}{dx} \left( \sqrt{x^2 + 4x + 1} \right) \quad \text{-- Ch. Rule} \\
& = \sec^2 \left( \sqrt{x^2 + 4x + 1} \right) \cdot \frac{1}{2 \sqrt{x^2 + 4x + 1}} \cdot \frac{d}{dx} (x^2 + 4x + 1) \quad \text{-- Ch. Rule again} \\
& = \sec^2 \left( \sqrt{x^2 + 4x + 1} \right) \cdot \frac{1}{2 \sqrt{x^2 + 4x + 1}} \cdot (2x + 4)
\end{align*}
\]

Suppose \( f''(x) \) exists at all points \( x \) of an interval \( I \). If \( f \) vanishes at three distinct points of \( I \), show that \( f'' \) must vanish at some point of \( I \).

Label the points \( x_1, x_2, x_3 \) in ascending order:

\[ x_1 < x_2 < x_3 \]

\[ f(x_1) = 0 = f(x_2). \quad \text{Rolle's Thm} \implies \text{There is a point } c_1 \text{ in } (x_1, x_2) \text{ so that } f'(c_1) = 0 \]

\[ f(x_2) = 0 = f(x_3). \quad \text{Rolle's Thm} \implies \text{there is a point } c_2 \text{ in } (x_1, x_3) \text{ so that } f''(c_2) = 0. \]

Now look at \( f'(x) \) on the interval \( [c_1, c_2] \).

\[ f'(c_1) = 0 = f'(c_2). \quad \text{Rolle's Thm} \implies \text{there is a point } c \text{ in } (c_1, c_2) \text{ so that } (f')'(c) = 0. \]

Thus \( f''(c) = 0 \).
Q2]...[20 points] Sketch the graph of the function

\[ f(x) = x^{1/3}(x - 4) \]

after you have answered the following questions. Make sure that your answers to these questions are visible/highlighted on your graph.

1. Find the intercepts of \( y = f(x) \) and determine the behavior of \( f(x) \) as \( x \to \infty \) and as \( x \to -\infty \).

   \[
   \begin{align*}
   \text{y-intercept:} & \quad y = f(0) = 0 \\
   \text{y-intercept:} & \quad \text{y-intercept} \\
   \text{x-intercepts:} & \quad x = 0, x = 4 \quad \text{for} \quad x^{1/3} = 0, x - 4 = 0
   \end{align*}
   \]

   \[
   \begin{align*}
   \lim_{x \to \infty} (x - 4) & = \infty, \quad \lim_{x \to \infty} x^{1/3} = \infty \\
   \text{Thus,} \quad & \lim_{x \to \infty} x^{1/3}(x - 4) = \infty. \\
   \lim_{x \to -\infty} (x - 4) & = -\infty, \quad \lim_{x \to -\infty} x^{1/3} = -\infty \\
   \text{Thus,} \quad & \lim_{x \to -\infty} x^{1/3}(x - 4) = -\infty.
   \end{align*}
   \]

2. Compute the derivative \( f'(x) \), and find all the critical points of \( f(x) \).

   \[
   \begin{align*}
   f(x) & = x^{1/3} - 4x^{1/3} \\
   f'(x) & = \frac{4}{3} x^{-2/3} - \frac{4}{3} x^{-2/3} \\
   & = \frac{4(x - 1)}{3x^{2/3}} \\
   \end{align*}
   \]

   Critical points are where \( f'(x) \) DNE and where \( f'(x) = 0 \).

   \[
   x = 0 \quad \text{Vertical Tangent Line at} \ x = 0. \\
   x = 1 \quad \text{Horizontal Tangent Line at} \ x = 1.
   \]

3. Determine the intervals where \( f(x) \) is increasing, and where \( f(x) \) is decreasing.

   \[
   \begin{array}{c|c|c|c}
   & (-\infty, 0) & (0, 1) & (1, \infty) \\
   \hline
   f'(x) & - & + & + \\
   f(x) & \downarrow & \uparrow & \uparrow \\
   \end{array}
   \]

   Note: \( \frac{4}{3x^{2/3}} \) is \( + \) for \( x \neq 0 \)

   Thus look at \( (x - 1) \) signs.

   Local min at \( x = 1 \), \( f(x) = 1^{1/3}(1 - 4) = -3 \)
4. Compute \( f''(x) \), and determine the intervals where \( f(x) \) is CCU, and where \( f(x) \) is CCD. Does the graph of \( f(x) \) have inflection points?

\[
\begin{align*}
  f'(x) &= \frac{4}{3} x^{\frac{1}{3}} - \frac{4}{3} x^{-\frac{2}{3}} \\
  f''(x) &= \frac{4}{9} x^{-\frac{2}{3}} + \frac{8}{9} x^{-\frac{5}{3}} \\
  &= \frac{4x + 8}{9x^{\frac{5}{3}}} \\
  &= \frac{2(x+2)}{9x^{\frac{5}{3}}}
\end{align*}
\]

\( f'' \) DNE at \( x = 0 \) & \( f'' = 0 \) at \( x = -2 \)

\( x = 0 \), \( x = -2 \) are both inflection points.

\[
\begin{array}{c|c|c|c}
  x & (-\infty, -2) & (-2, 0) & (0, \infty) \\
  \hline
  f'' & + & = & + \\
  \hline
  f & CCU & CCD & CCU \\
\end{array}
\]

Now sketch the graph \( y = f(x) \):

\[\text{Decreasing} \quad \rightarrow \quad \text{Increasing} \]
Q3]...[12 points] Show that if it is possible to draw three normal lines from the point \((a, 0)\) to the parabola \(x = y^2\), then \(a\) must be greater than \(\frac{1}{2}\).

\[x = y^2\]
\[\frac{dy}{dx} = \frac{dy^2}{dx} = 2y \quad \text{Implicit differentiation}
\]

Tangent slope: \(y' = \frac{1}{y}\)

Normal slope: \(-2y\)

Also, \(\frac{a-y}{a-x} = \frac{-y}{a-y^2}\)

Get \(\frac{a-y}{a-y^2} = 2(y)\)

\[\Rightarrow 1 = 2(a-y^2) \quad \Rightarrow \begin{align*}
    a &= \frac{1}{2} + y^2 \\
    a &> \frac{1}{2}
\end{align*} \]

Need \(a > \frac{1}{2}\)!

One of the three normals above is the \(x\)-axis. Find the value of \(a\) for which the other two normals are perpendicular to each other.

Want \((-2y)(2y) = -1\) \ \perp \ \text{lines}

\[-4y^2 = -1\]

\[y^2 = \frac{1}{4} \quad y = \pm \frac{1}{2}\]

\[\Rightarrow a = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}\]

Find the value(s) of \(a\) for which the other two normals intersect at an angle of \(\pi/3\).

Two cases:

\[\begin{align*}
    \text{Slope} &= -\frac{1}{\sqrt{3}} = -\frac{1}{2}y \\
    y &= \frac{1}{2\sqrt{3}} \\
    y^2 &= \frac{1}{12} \\
    a &= \frac{1}{2} + \frac{1}{12} = \frac{7}{12}
\end{align*}\]
Q4]... [8 points] Find the absolute maximum and the absolute minimum of the function

\[ f(x) = x + \frac{4}{x} \]

on the interval \([1, 6]\).

1. **End pts.** \(x = 1\) \(x = 6\)

2. **Critical pts.**
   \[ f'(x) = 1 - \frac{4}{x^2} \quad \text{always exists on} \quad [1, 6] \]
   
   \[ f'(x) = 0 \Rightarrow 1 = \frac{4}{x^2} \quad 4 = x^2 \quad x = \pm 2 \]
   
   \[ x = 2 \quad \text{only one} \quad x \in [1, 6]. \]

3. **Outputs.**
   
   \[ f(1) = 1 + \frac{4}{1} = 5 \]
   
   \[ f(2) = 2 + \frac{4}{2} = 2 + 2 = 4 \]
   
   \[ f(6) = 6 + \frac{4}{6} = 6 + \frac{2}{3} = \frac{20}{3} \]

   - **Absolute Min.**
   - **Absolute Max.**