Q1... [15 points] For each of the following, say if the statement is true or false.

(a) If \( f(x) \) and \( g(x) \) each have second derivatives, then

\[
\frac{d^2(fg)}{dx^2} = \frac{d^2f}{dx^2}g + f\frac{d^2g}{dx^2}
\]

**FALSE**

\((fg)'' = (fg')' = (f'g + fg')' \quad \ldots \text{product Rule}
\]

\[
= (f'g)' + (fg')' \quad \ldots \text{sum Rule}
\]

\[
= f''g + f'g' + fg' + f'g' \quad \text{Product Rule (not same as right hand side above!)}
\]

(b) If the position of a particle at time \( t \) is given by \( x(t) = t^3 - 3t^2 \), then the particle is decelerating (slowing down) during the interval from time \( t = 0 \) until time \( t = 1 \).

Velocity \( = x'(t) = 3t^2 - 6t \); acceleration \( = x''(t) = 6t - 6 \)

**TRUE**

\( x''(0) = -6 \quad \Rightarrow x''(t) \text{ is } 0 \text{ for } t \text{ between } 0 \text{ and } 1 \)

(c) If \( f(x) \) is differentiable at the point \( a \) then

\[
\lim_{h \to 0} \frac{f(a) - f(a-h)}{h} = f'(a)
\]

**FALSE**

\[
\lim_{h \to 0} \frac{f(a) - f(a-h)}{h} = \lim_{h \to 0} \frac{f(a-h) - f(a)}{-h} = \lim_{h \to 0} \left( \frac{f(a+h) - f(a)}{h} \right) = f'(a)
\]

\[= -f'(a).\]

(d) The piecewise defined function \( y \) is continuous at 0

\[
y = \begin{cases} 
  x \sin(1/x) & \text{when } x < 0 \\
  0 & \text{when } x = 0 \\
  x^2 \cos(1/x) & \text{when } x > 0 
\end{cases}
\]

By Squeeze Theorem: \( \lim_{x \to 0^-} (y) = 0 \) and \( \lim_{x \to 0^+} (y) = 0 \)

Both of these are equal & are equal to \( y(0) \).

(e) **TRUE**

\[
\lim_{x \to 3} \frac{x^{10} - 3^{10}}{x - 3} = 10(3^9)
\]

\[
\lim_{x \to 3} \frac{x^{10} - 3^{10}}{x - 3} = \frac{d}{dx} x^{10} \bigg|_{x=3} = 10 x^9 \bigg|_{x=3} = 10(3)^9
\]

Defintion of \( \frac{dy}{dx} \). Power Rule.
Q2]...[15 points] Write down the values of the following two limits (you do not have to give proofs).

\[ \lim_{x \to 0} \frac{\sin(x)}{x} = 1 \quad \boxed{(1)} \]

\[ \lim_{x \to 0} \frac{1 - \cos(x)}{x} = 0 \quad \boxed{(2)} \]

Write out the angle addition formula for the cosine function.

\[ \cos(A + B) = \cos(A) \cos(B) - \sin(A) \sin(B) \quad \boxed{(3)} \]

Compute the derivative of \( \cos(x) \) at the point \( a \) using the limit of the difference quotient definition of derivative. Show all your work.

\[ \frac{d}{dx} \cos(x) \bigg|_{x=a} = \lim_{h \to 0} \left( \frac{\cos(a+h) - \cos(a)}{h} \right) \]

by (3) \[ = \lim_{h \to 0} \left( \frac{\cos(a) \cos(h) - \sin(a) \sin(h) - \cos(a)}{h} \right) \]

\[ = \lim_{h \to 0} \left[ \cos(a) \left( \frac{\cos(h) - 1}{h} \right) - \sin(a) \left( \frac{\sin(h)}{h} \right) \right] \]

Limit laws \[ = \cos(a) \lim_{h \to 0} \left( \frac{\cos(h) - 1}{h} \right) - \sin(a) \lim_{h \to 0} \left( \frac{\sin(h)}{h} \right) \]

by (1) \[ = \cos(a) \cdot 0 - \sin(a) \cdot 1 = -\sin a \]
Q3]...[8 points] Verify that the graphs of \( y = x^2 \) and \( y = \frac{1}{\sqrt{x}} \) intersect at the point (1, 1).

\[
y = x^2.
\]
When \( x = 1 \)
\[
y = (1)^2 = 1
\]

(1, 1) Graph contains

Both graphs contain (1, 1), so they intersect at (1, 1).

\[
y = \frac{1}{\sqrt{x}},
\]
When \( x = 1 \)
\[
y = \frac{1}{\sqrt{1}} = 1 = 1
\]

(1, 1) Graph contains (1, 1).

Show that these graphs are perpendicular at the intersection point (1, 1); that is, show that their tangent lines at the point (1, 1) are perpendicular.

\[
y = x^2
\]
Tangent line slope
\[
= \frac{dy}{dx} \bigg|_{x=1}
\]
\[
= 2x \bigg|_{x=1}
\]
\[
= 2(1) = 2
\]

\[
y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}
\]
Tangent line slope
\[
= \frac{dy}{dx} \bigg|_{x=1}
\]
\[
= -\frac{1}{2}x^{-\frac{3}{2}} \bigg|_{x=1}
\]
\[
= -\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1}}
\]
\[
= -\frac{1}{2}
\]

Product of slopes = \( (2) \times \left( -\frac{1}{2} \right) = -1 \)

\( \Rightarrow \) Lines are \( \perp \)

\( \Rightarrow \) Curves intersect perpendicularly.
Q4...[12 points] Compute the derivatives $y'$ of the following functions. Write down the names of the differentiation rules that you used in each case.

**Quotient Rule:**

$$y = \frac{\sin(x) + 4x + 3}{x^8 - 5x}$$

$$y' = \frac{(\cos(x) + 4)(x^8 - 5x) - (\sin(x) + 4x + 3)(8x^7 - 5)}{(x^8 - 5x)^2}$$

(also sum & power rules).

**Product Rule:**

$$y = (\sqrt{x} + x + 7)(x^8 - 5x + 3)$$

$$y' = \left(\frac{1}{2\sqrt{x}} + 1\right)(x^8 - 5x + 3) + (\sqrt{x} + x + 7)(8x^7 - 5)$$

(also sum & power rules)