...[15 points] Compute the following derivatives.

Find \( y' \) in terms of \( x \) and \( y \) where

\[
x^2 + 4y^2 = 9
\]

**Implicit Diff:**

\[
\frac{d}{dx} (x^2 + 4y^2) = \frac{d}{dx} 9
\]

\[
2x + 8y \frac{dy}{dx} = 0
\]

\[
\frac{dy}{dx} = -\frac{2x}{8y}
\]

Find \( f^{(20)} \) where

\[
f(x) = 5x^{21} - 4x^{19} + \sin(7x + 3)
\]

\[
f'(x) = 5(21)x^{20} - 4(19)x^{18} + 7 \cos(7x + 3)
\]

\[
\vdots
\]

\[
f^{(20)}(x) = 5(21)(20)\ldots(3)x^{1} - 0 + 7^{20} \sin(7x + 3)
\]

\[
f^{(20)}(x) = 5(21!)x + 7^{20} \sin(7x + 3)
\]

Find \( g'(x) \) where

\[
g(x) = \int_{\tan(x)}^{5} \sqrt{1 + t^2} \, dt
\]

\[
g(x) = -\int_{5}^{\tan(x)} \sqrt{1 + t^2} \, dt
\]

\[
g'(x) = -\sqrt{1 + \tan^2(x)} \cdot \sec^2(x)
\]

\[
(Fund Thm^m
\]

\[
+ \text{Chain Rule}
\]
...[10 points] Evaluate the following integrals (one indefinite and one definite).

\[ \int x \sec^2(x^2 + 1) \, dx \]

\[ \text{Subst}^2: \quad u = x^2 + 1 \quad \text{du} = 2x \, dx \]
\[ \frac{du}{2} = x \, dx \]

\[ \int = \int \sec^2(u) \, \frac{du}{2} = \frac{1}{2} \tan(u) + C \]
\[ = \frac{1}{2} \tan(x^2 + 1) + C \]

\[ \int_0^2 (5 - 2x) \sqrt{4 - x^2} \, dx \]

\[ = 5 \int_0^2 \sqrt{4 - x^2} \, dx + \int_0^2 (-2x) \sqrt{4 - x^2} \, dx \]

\[ \text{Subst}^2: \quad u = 4 - x^2 \quad \text{du} = -2x \, dx \]

\[ = 5 \left( \frac{1}{4} \pi (2)^2 \right) + \int_{4-0^2}^{4-2^2} u^{1/2} \, du \]

\[ = 5 \pi + \left[ \int_4^{\infty} u^{1/2} \, du \right] = 5 \pi + \left[ \frac{2}{3} u^{3/2} \right]_4^{\infty} \]
\[ = 5 \pi - \frac{16}{3} \]
Q3]...[10 points] A highway patrol plane flies 3 mi vertically above a level straight road. The plane flies horizontally (parallel to the road) at 120 mi/hr. The pilot sees an oncoming car on the road. The planes radar determines that at the instant the line-of-sight distance from the plane to the car is 5 mi, this distance is decreasing at a rate of 160 mi/hr. Determine the speed of the car along the highway at this instant.

\[ \frac{dh}{dt} = 120 \]

\[ h = \text{line-of-sight distance} \]

\[ h^2 = x^2 + 3^2 \]

\[ \frac{d}{dt} \]

\[ 2 \frac{h \cdot dh}{dt} = 2x \frac{dx}{dt} + 0 \]

\[ 2(5)(-160) = 2(4) \frac{dx}{dt} \]

\[ \frac{dx}{dt} = -\frac{5}{4}(160) \]

\[ = -200 \text{ mi/hr} \]

\[ -200 \text{ mi/hr} = - \left( \frac{dp}{dt} + \frac{dc}{dt} \right) \]

\[ \frac{\pi}{120} \]

\[ \Rightarrow \frac{dc}{dt} = 200 - 120 = 80 \text{ mi/hr} \]
Q4] [10 points] Draw the region $R$ in the plane which is bounded above by the line $y = 1$ and bounded below by the parabola $y = x^2$.

Use the SHELL method to write down an integral for the volume obtained by rotating the region $R$ about the line $x = 2$. You do not have to evaluate the integral.

$$V_{\text{shell}} = \int_{-1}^{1} 2\pi (\text{radius})(\text{height}) \, dx$$

$$= \int_{-1}^{1} 2\pi (2-x)(1-x^2) \, dx$$

Use the WASHER (disk with hole) method to write down an integral for the volume obtained by rotating the region $R$ about the line $x = 2$. You do not have to evaluate the integral.

$$V_{\text{washer}} = \int_{0}^{1} \pi \left[ (r_{\text{out}})^2 - (r_{\text{inn}})^2 \right] \, dy$$

$$= \int_{0}^{1} \pi \left[ (2 + \sqrt{y})^2 - (2 - \sqrt{y})^2 \right] \, dy$$
Q5...[10 points]  Find the maximum volume of a right circular cone which can be inscribed inside of a sphere of radius $R$.

\[ \text{Cross Section. (Rotate about } y\text{-axis to get the 3-dimensional picture.)} \]

\[ \text{Base of Cone crosses } y\text{-axis at } y \Rightarrow \]
\[ (\text{height of cone}) = (R - y). \]
\[ (\text{radius of cone base}) = x = \sqrt{R^2 - y^2} \]

\[ V_{\text{cone}} = \frac{1}{3} \pi x^2 \cdot h \]
\[ = \frac{1}{3} \pi (R^2 - y^2) (R - y) \]

\[ V(y) = \frac{\pi}{3} \left( R^3 - R^2 y - R y^2 + y^3 \right) \]

\[ \frac{dV}{dy} = 0 \Rightarrow \frac{\pi}{3} (0 - R^2 - 2Ry + 3y^2) = 0 \]

\[ 3y^2 + 2Ry - R^2 = 0 \]
\[ (3y + R)(y - R) = 0 \]

\[ y = -\frac{R}{3}, \quad y = R \]

\[ \text{Max } V = \frac{\pi}{3} \left( R^2 - \left(\frac{R}{3}\right)^2 \right) \left( R - \left(\frac{R}{3}\right) \right) \]
\[ = \frac{\pi}{3} \left( \frac{8R^3}{9} \right) \left( \frac{4R}{3} \right) = \frac{32\pi R^3}{81} \]
Q6) [10 points] Write down the formula for Newton’s method of approximating roots of \( f(x) \).

\[
    x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}
\]

A classical algorithm for approximating the square root of a number \( A \) is the divide and average method. You have an estimate \( x_1 \) for \( \sqrt{A} \). Now divide it into \( A \). If the original estimate \( x_1 \) is larger (resp. smaller) than \( \sqrt{A} \), then \( \frac{A}{x_1} \) is smaller (resp. larger) than \( \sqrt{A} \). So the average of the two

\[
    \frac{1}{2} \left( x_1 + \frac{A}{x_1} \right)
\]

is taken to be the next approximation. Denote this by \( x_2 \) and repeat the process starting with \( x_2 \).

Show that the divide and average method is exactly the algorithm that Newton’s method gives for approximating the root \( \sqrt{A} \) of the function \( f(x) = x^2 - A \).

\[
    f(x) = x^2 - A
\]
\[
    f'(x) = 2x
\]

Newton \( \Rightarrow \) \( x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \)

\[
    = x_1 - \frac{x_1^2 - A}{2x_1}
\]

\[
    = x_1 - \frac{x_1}{2} + \frac{A}{2x_1}
\]

\[
    = \frac{x_1}{2} + \frac{A}{2x_1}
\]

\[
    = \frac{1}{2} \left( x_1 + \frac{A}{x_1} \right)
\]

which is same expression.
Q7) [15 points] Express the following limits as definite integrals.

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{1}{n} \sqrt{\frac{i}{n}} = \int_{0}^{1} \sqrt{x} \, dx
\]

\[
\lim_{n \to \infty} \sum_{i=1}^{n} \frac{n}{n^2 + i^2} = \int_{0}^{1} \frac{1}{1 + x^2} \, dx
\]

If \( f(x) \) is differentiable and \( f'(2) = 7 \), what is the value of the following limit?

\[
\lim_{x \to 2} \frac{f(x) - f(2)}{\sqrt{x} - \sqrt{2}}
\]

\[
= \lim_{x \to 2} \left( \frac{f(x) - f(2)}{\sqrt{x} - \sqrt{2}} \cdot \frac{\sqrt{x} + \sqrt{2}}{\sqrt{x} + \sqrt{2}} \right)
\]

\[
= \lim_{x \to 2} \left( \frac{f(x) - f(2)}{x - 2} \right) \left( \sqrt{x} + \sqrt{2} \right)
\]

\[
= \lim_{x \to 2} \left( \frac{f(x) - f(2)}{x - 2} \right) \cdot \lim_{x \to 2} \left( \sqrt{x} + \sqrt{2} \right)\]

\[
= f'(2) \left( \sqrt{2} + \sqrt{2} \right) = 7 \left( 2 \sqrt{2} \right) = 14 \sqrt{2}
\]
Q8]...[10 points] Find the intervals where \( f(x) \) is increasing, decreasing, concave up, concave down. Find local maxima, minima and points of inflection. You do not have to sketch the graph of \( f \).

\[
f(x) = \frac{x}{x^2 + 3}
\]

\[
f'(x) = \frac{1(x^2 + 3) - x(2x)}{(x^2 + 3)^2} = \frac{3 - x^2}{(x^2 + 3)^2}
\]

\[
\begin{array}{c|ccc}
& (-\infty, -\sqrt{3}) & (-\sqrt{3}, \sqrt{3}) & (\sqrt{3}, \infty) \\
3 - x^2 & - & + & - \\
\hline
f'(x) & - & + & - \\
\hline
f(x) & \downarrow & \uparrow & \downarrow \\
\end{array}
\]

\(-\sqrt{3}: \text{local Min}\) \quad \(\sqrt{3}: \text{local Max}\)

\[
f''(x) = \frac{(-2x)(x^2 + 3)^2 - (3 - x^2)2(x^2 + 3)(2x)}{(x^2 + 3)^4}
\]

\[
= \frac{-2x^3 - 6x - 12x + 4x^3}{(x^2 + 3)^3} = \frac{2x^3 - 18x}{(x^2 + 3)^2} = \frac{2x(x^2 - 9)}{(x^2 + 3)^2}
\]

\[
f''(x) = 0
\]

\(x = 0, \quad x = \pm 3\)

\[
\begin{array}{c|cccc}
& (-\infty, -3) & (-3, 0) & (0, 3) & (3, \infty) \\
2x(x^2 - 9) & - & + & - & + \\
\hline
f & \text{CCD} & \text{CCU} & \text{CCD} & \text{CCU} \\
\end{array}
\]

\(-3, 0, 3 \text{ are all inflection points.}\)
Q9]...[10 points] Newton's law of motion states that

\[ F = m \frac{dv}{dt} \]

where \( F \) is a force acting on a particle of mass \( m \), and \( \frac{dv}{dt} \) is the acceleration of the particle.

Use Newton's law of motion to derive a law of conservation of energy for a particle of mass \( m \) moving from point \( A \) to point \( B \) along the \( x \)-axis under a spring force \( F = -kx \).

\[-kx = m \frac{dv}{dt} = m \frac{dv}{dx} \frac{dx}{dt} \]

\[= m \frac{dv}{dx} \quad \text{--- Ch. Rule} \]

\[= MV \frac{dv}{dx} \quad \text{--- velocity, } v = \frac{dx}{dt} \]

Now take \( \int_A^B \) of both sides.

\[ \int_A^B -kx \, dx = \int_A^B MV \frac{dv}{dx} \, dx \]

\[-\frac{kx^2}{2} \bigg|_A^B \quad \text{--- power rule} \]

\[= -\frac{kB^2}{2} + \frac{kA^2}{2} \]

\[= \int_{V_A}^{V_B} MV \, dv \quad \text{--- subst } v \]

\[= \frac{MV_B^2}{2} - \frac{MV_A^2}{2} \quad \text{--- power rule} \]

\[\frac{kA^2}{2} - \frac{kB^2}{2} = \frac{1}{2} MV_B^2 - \frac{1}{2} MV_A^2 \]

\[\implies \frac{kA^2}{2} + \frac{kA^2}{2} = \frac{kB^2}{2} + \frac{1}{2} MV_B^2 \]

\[
\text{Conservation of energy}
\]