MID II — SOLUTIONS

Q1]...[10 points] Prove that the following are true for sets $A$and B.

$$(A \cup B) \cap \overline{(A \cap B)} = (A \cap \overline{B}) \cup (B \cap \overline{A})$$

$$(A \cup B) \cap \overline{(A \cap B)} = (A \cup B) \cap (\overline{A} \cup \overline{B}) \quad \text{---- de Morgan}$$

$$= (A \cap (\overline{A} \cup \overline{B})) \cup (B \cap (\overline{A} \cup \overline{B})) \quad \text{--- distrib } \cap \text{ over } \cup$$

$$= (A \cap \overline{A}) \cup (A \cap \overline{B}) \cup (B \cap \overline{A}) \cup (B \cap \overline{B}) \quad \text{.... distrib } \cap \text{ over } \cup$$

$$= \emptyset \cup (A \cap \overline{B}) \cup (B \cap \overline{A}) \cup \emptyset \quad \text{----} \quad X \cap \overline{X} = \emptyset$$

$$= (A \cap \overline{B}) \cup (B \cap \overline{A}) \quad \text{------} \quad X \cup \emptyset = X$$

$$A \cup (B \setminus A) = A \cup B$$

$$A \cup (B \setminus A) = A \cup (B \cap \overline{A}) \quad \text{------} \quad A \setminus B = A \cap \overline{B}$$

$$= (A \cup B) \cap (A \cup \overline{A}) \quad \text{---- distrib } \cup \text{ over } \cap$$

$$= (A \cup B) \cap \mathcal{U} \quad \text{----} \quad X \cup \overline{X} = \mathcal{U}$$

$$= (A \cup B) \quad \text{------} \quad X \cap \mathcal{U} = X$$

$\mathcal{U}$ = 'universe'

Q2]... **[10 points]** Suppose that $f : X \to Y$ is a function, and that $A \subset X$ and $B \subset X$. Prove that $f(A \cap B) \subset f(A) \cap f(B)$.

$y \in f(A \cap B) \Rightarrow y = f(x)$ for some $x \in A \cap B$

Thus $y = f(x)$ for some $x \in A$ --- since $x \in A \cap B \subseteq A$

Also $y = f(x)$ for some $x \in B$ ---- since $x \in A \cap B \subseteq B$

Thus $y \in f(A)$ and $y \in f(B)$

$\Rightarrow y \in f(A) \cap f(B)$.

$\Rightarrow \boxed{f(A \cap B) \subseteq f(A) \cap f(B)}$ — I

Give an example to show that $f(A \cap B)$ need not be equal to $f(A) \cap f(B)$.

$f : \mathbb{R} \to \mathbb{R} : x \mapsto x^2$

$A = (-\infty, 1]$ $B = [-1, \infty)$ $A \cap B = [-1, 1]$

$f(A) = [0, \infty)$ $f(B) = [0, \infty)$ $f(A \cap B) = [0, 1]$ — 2

$f(A) \cap f(B) = [0, \infty) \cap [0, \infty) = [0, \infty)$ — 1

$\boxed{[0,1] \subsetneqq [0, \infty)}$

Prove that $f(A \cap B) = f(A) \cap f(B)$ under the additional assumption that f is an injective map.

We've already seen that $f(A \cap B) \subseteq f(A) \cap f(B)$ holds in general.
Now assume f is injective.
Given $y \in f(A) \cap f(B)$. Then $y = f(a)$ for some $a \in A$ and
$y = f(b)$ for some $b \in B$.

$\Rightarrow f(a) = y = f(b)$

$\Rightarrow a = b$ ---- since f is injective.

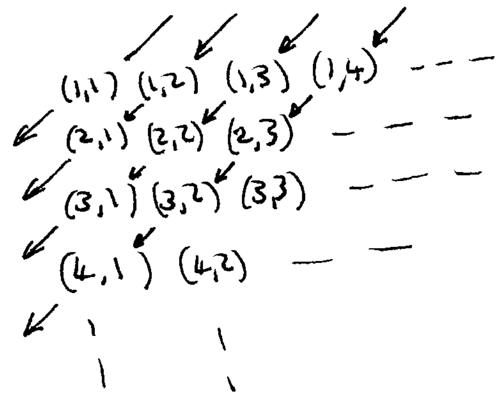
Thus $a = b \in B$ and $b = a \in A \Rightarrow a = b \in A \cap B$.

Thus $y = f(a) = f(b) \in f(A \cap B)$. Therefore $\boxed{f(A) \cap f(B) \subseteq f(A \cap B)}$ — II

Combining I & II gives equality.

Q3]... [10 points] For each of the following pairs of sets, say if they have the same cardinality or not. Give arguments (proofs) to justify your answers in each case.

$\mathbb{Z}^+$ and $\mathbb{Z}^+ \times \mathbb{Z}^+$. $\boxed{\text{Yes}}$ SAME CARDINALITY



SKETCH: (a) Arrange elements of $\mathbb{Z}^+ \times \mathbb{Z}^+$ as shown

(b) Now list elements as shown, work through diagonals from left to right, & work down a diagonal along arrows.

(c) This list gives a bijective map with $\mathbb{Z}^+$

$$\begin{array}{cccccccc} (1,1) & (1,2) & (2,1) & (1,3) & (2,2) & (3,1) & \cdots \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \updownarrow & \\ 1 & 2 & 3 & 4 & 5 & 6 & \cdots \end{array}$$

More details on (b) ... Don't need to refer to "pictures"!!

(i) List elements of $\mathbb{Z}^+ \times \mathbb{Z}^+$ according to ~~i~~ nondecreasing sum of coordinates

sum = 2, 3, 4, ----

(ii) Within a particular sum, $(n+1)$ say, list elements by increasing 1st coordinate: $(1,n), (2,n-1), (3,n-2), \ldots, (n,1)$.

$\mathbb{Z}^+$ and $(0,1) = \{x \in \mathbb{R} | 0 < x < 1\}$.

$\boxed{\text{No}}$ Different CARDINALITIES

We will prove $\nexists\ f : \mathbb{Z}^+ \to (0,1)$ which is bijective (surjective!)

Argue by contradiction. Suppose $\exists$ bijection $f : \mathbb{Z}^+ \to (0,1)$

$$\left.\begin{aligned} f(1) &= 0.\boxed{a_{11}}\, a_{12}\, a_{13} \cdots \\ f(2) &= 0.\, a_{21}\, \boxed{a_{22}}\, a_{23} \cdots \\ f(3) &= 0.\, a_{31}\, a_{32}\, \boxed{a_{33}} \cdots \\ &\ \vdots \qquad\qquad\quad \ddots \end{aligned}\right] \text{ where } a_{ij} \in \{0,1,\ldots,9\} \text{ for all } i,j.$$

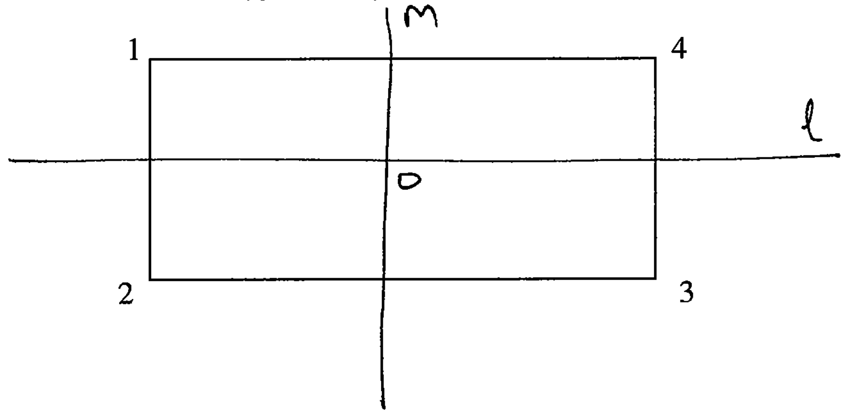
Form a new number $x = 0.b_1 b_2 b_3 \cdots \in (0,1)$ as follows.

$b_1 \neq a_{11},\ b_2 \neq a_{22},\ \ldots,\ b_n \in \{0,1,\ldots,9\} \setminus \{a_{nn}\}, \ldots$

Clearly, $x \in (0,1)$. By construction $x \notin$ Image of f. $\Rightarrow$ f not onto. $\Rightarrow\!\Leftarrow$.

ensure $b_i \neq 0\ \forall i$, and $b_i \neq 9\ \forall i$

Q4]... **[10 points]** How many symmetries does the rectangle below have? Describe them, and write down a composition table for them. Also, use the vertex labeling shown to identify each symmetry with an element of the set of permutations Perm({1, 2, 3, 4}).



(Reflection in line l) $= l \longleftrightarrow (12)(34)$

(Reflection in line m) $= m \longleftrightarrow (14)(23)$

(180° rot$^{\underline{n}}$ about O) $= R \longleftrightarrow (13)(24)$

$\mathbb{1}_{\mathbb{R}^2} \longleftrightarrow \mathbb{1} = (1)(2)(3)(4)$

There are just 4 symmetries.

Composition Table.

$\circ$	$\mathbb{1}$	R	l	m
$\mathbb{1}$	$\mathbb{1}$	R	l	m
R	R	$\mathbb{1}$	m	l
l	l	m	$\mathbb{1}$	R
m	m	l	R	$\mathbb{1}$

Q5]... [10 points] True/False. Give *reasons* for your answers. In these questions, capital letters A, B, C, X, Y denotes sets, and small letters are used to denote either functions (f, g) or elements of sets, y.

1. If $|A| = 3$ and $|B| = 4$, then $|A \cup B|$ must be equal to 7.

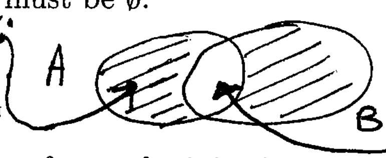
$\boxed{\text{FALSE}}$ $|A \cup B| = |A| + |B| - |A \cap B| = 3 + 4 - |A \cap B| = 7 - |A \cap B| \neq 7$ If $A \cap B \neq \emptyset$.

2. If $A \cup C = B \cup C$, then A must equal B.

$\boxed{\text{FALSE}}$ eg. A & B can be disjoint sets which are both subsets of C. $A \cup C = C = B \cup C$ but $A \neq B$.

3. If $A \oplus B = B$, then A must be $\emptyset$.

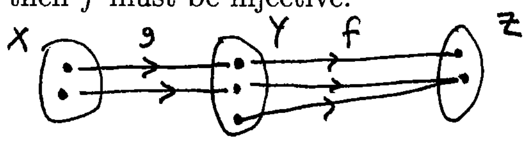
$\boxed{\text{TRUE}}$ This must be $\emptyset$, otherwise $A \oplus B$ would contain elements not in B.



This, $(A \cap B)$, must be $\emptyset$; otherwise B would contain elements which are not in $A \oplus B$.

4. If $f \circ g$ is injective, then f must be injective.

$\boxed{\text{FALSE}}$ eg.



5. If $f \circ g$ is surjective, then f must be surjective.

$X \xrightarrow{g} Y \xrightarrow{f} Z$

$\boxed{\text{TRUE}}$ ↳ Given $z \in Z$ $\exists$ $x \in X$ so that $(f \circ g)(x) = z$.
Thus $f(g(x)) = z$, But $g(x) \in Y$. So we've found $g(x) \in Y$ so that $f(g(x)) = z$
$\Rightarrow$ f is surjective.

6. If $f : X \to Y$ is injective and $y \in Y$, then $|f^{-1}(\{y\})|$ must be 1.

$\boxed{\text{FALSE}}$ if f is not surjective and $y \in Y$ is not in the image of f, we have $|f^{-1}(\{y\})| = |\emptyset| = 0 \neq 1$.

7. The product of permutations $(1234)(234)$ is equal to (1243).

$\boxed{\text{TRUE}}$

$1 \mapsto 1 \mapsto 2$
$2 \mapsto 3 \mapsto 4$
$3 \mapsto 4 \mapsto 1$
$4 \mapsto 2 \mapsto 3$

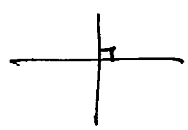
$1 \to 2 \to 4 \to 3$ (1243)

8. The union of two disjoint countably infinite sets, is again countably infinite.

$\boxed{\text{TRUE}}$ List A ; $a_1, a_2, \ldots$
List B ; $b_1, b_2, \ldots$
Now list Union $A \cup B$: $a_1, b_1, a_2, b_2, \ldots$

9. The composition of reflections in two perpendicular lines in the plane is equal to a 90° rotation about their intersection point.

$\boxed{\text{FALSE}}$



it's a 180° Rotation about intersection point.

10. If $|A| = 3$, $|B| = |C| = 5$, $|A \cap B| = 2$, $|B \cap C| = 3$, and $|A \cap C| = |A \cap B \cap C| = 1$, then $|A \cup B \cup C| = 8$.

$\boxed{\text{TRUE}}$ $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$
$= 8$.