Q1]... [10 points] Evaluate the following two limits, showing all your work.

\[
\lim_{t \to 1} \frac{t^2 - 1}{t^2 - 2t + 1} = \lim_{t \to 1} \frac{(t-1)(t+1)}{(t-1)(t-1)} = \lim_{t \to 1} \left( \frac{t+1}{t-1} \right)
\]

\(\xrightarrow{t \to 1^+}\) numerator \(\to 2\) \(\xrightarrow{t \to 1^-}\) denominator \(\to 0^-\) \(\Rightarrow\) fraction \(\to -\infty\)

\(\xrightarrow{t \to 1^+}\) denominator \(\to 0^+\) \(\Rightarrow\) fraction \(\to +\infty\)

Therefore \(\text{Limit DNE (Does not exist)}\).

\[
\lim_{x \to 8} \frac{x^{2/3} - 4}{x^{1/3} - 2} = \lim_{x \to 8} \left( \frac{\left(x^{1/3}\right)^2 - 4}{x^{1/3} - 2} \right) = \lim_{x \to 8} \left( \frac{x^{1/3} - 2}{x^{1/3} - 2} \left(x^{1/3} + 2\right) \right)
\]

\(= \lim_{x \to 8} \left( x^{1/3} + 2 \right) \)

\(= 8^{1/3} + 2 = 2 + 2 = 4\)

**Mid I - Solutions**
Q2] [10 points] Evaluate the following limits, showing all your work.

\[
\lim_{x \to \infty} \left( \sqrt{x^2 + 2x} - \sqrt{x^2 - 2x} \right)
\]

\[
= \lim_{x \to \infty} \left( \frac{\sqrt{x^2 + 2x} - \sqrt{x^2 - 2x}}{1} \right) \left( \frac{\sqrt{x^2 + 2x} + \sqrt{x^2 - 2x}}{\sqrt{x^2 + 2x} + \sqrt{x^2 - 2x}} \right)
\]

\[
= \lim_{x \to \infty} \frac{x^2 + 2x - (x^2 - 2x)}{\sqrt{x^2 + 2x} + \sqrt{x^2 - 2x}} = \lim_{x \to \infty} \frac{4x}{\sqrt{x^2 + 2x} + \sqrt{x^2 - 2x}}
\]

\[
= \lim_{x \to \infty} \left( \frac{4}{\sqrt{1 + \frac{2}{x}} + \sqrt{1 - \frac{2}{x}}} \right) = \frac{4}{\sqrt{1} + \sqrt{1}} = \frac{4}{2} = 2
\]

\[
\lim_{t \to 2^+} \frac{t}{(2-t)^3}
\]

\[
= \lim_{t \to 2^+} \frac{\frac{t}{t}}{(2-t)^3} = \frac{t}{t} \cdot \frac{1}{(2-t)^3}
\]

Since

\[
\lim_{t \to 2^+} (2-t)^3 \to 0^-
\]

\[
\Rightarrow \quad \frac{t}{(2-t)^3} \to 0^-
\]

\[
\Rightarrow \quad \text{also} \quad t \to 2
\]

\[
\Rightarrow \quad \text{radlim} \quad \to \quad -\infty
\]
Q3]...[10 points] Find the equation of the tangent line to the graph of the function \( y = (x - 1)^{1/3} \) at the point (2, 1). Show all your work carefully. Note that the 1/3 is an exponent (and so denotes a cube root). You are only to work with ideas and techniques from chapters 1 and 2 of the book (do not use quick methods or special rules).

\[ \text{Equation of line } \quad (y - y_1) = m (x - x_1) \]

Point: \((x_1, y_1)\)

Slope: \(m\)

1st need slope:

\[ \lim_{h \to 0} \left( \frac{(x+h)^{1/3} - (x-1)^{1/3}}{h} \right) \]

\[ = \lim_{h \to 0} \left( \frac{(h+1)^{1/3} - 1}{h} \right) \]

\[ = \lim_{h \to 0} \left( \frac{(h+1)^{1/3} - 1}{h} \right) \left( \frac{(h+1)^{2/3} + (h+1)^{1/3} + 1}{(h+1)^{2/3} + (h+1)^{1/3} + 1} \right) \]

\[ = \lim_{h \to 0} \left( \frac{(h+1)^{1/3} - 1}{h} \right) \left( \frac{1}{(h+1)^{1/3} + (h+1)^{1/3} + 1} \right) \]

\[ = \lim_{h \to 0} \frac{1}{(h+1)^{1/3} + (h+1)^{1/3} + 1} = \frac{1}{3} \]

Now, point: (2, 1) so equation is:

\[ (y - 1) = \frac{1}{3} (x - 2) \]
Q4]...[10 points] Find the value of the constant c which makes the function \( f \) below continuous. Show how you obtained your answer.

\[
f(x) = \begin{cases} 
  x^2 + c & \text{if } x \geq 1 \\
  7x - 1 & \text{if } x < 1
\end{cases}
\]

continuous at \( x = 1 \)? Justify your answer.

\[
\lim_{x \to 1^+} f(x) = (1)^2 + c = 1 + c \\
\lim_{x \to 1^-} f(x) = \lim_{x \to 1} (7x - 1) = 7(1) - 1 = 6
\]

Therefore, \( 1 + c = 6 \Rightarrow c = 5 \)

Is the function 

\[
g(x) = \begin{cases} 
  x \sin \left( \frac{1}{x} \right) & \text{if } x \neq 0 \\
  0 & \text{if } x = 0
\end{cases}
\]

continuous at \( x = 0 \)? Justify your answer.

Yes... since 

\[-(1) \leq x \sin \left( \frac{1}{x} \right) \leq |1| \]

\[
\lim_{x \to 0} \left( x \sin \left( \frac{1}{x} \right) \right) = 0 \quad \text{by Squeeze Thm.}
\]

But this agrees with \( g(0) \).

Thus \( \lim_{x \to 0} g(x) = 0 = g(0) \) & so \( g(x) \) is indeed continuous at \( 0 \).
Q5]...[10 points] Write down an expression (no proof necessary) for the sine of the sum of two angles in terms of the sines and cosines of the two angles.

\[ \sin(x+h) = \sin(x) \cos(h) + \cos(x) \sin(h) \]

Show how the expression above is used in showing that the sine function is continuous at an arbitrary point \( x \).

We want to show:

\[ \lim_{t \to x} \sin(t) = \sin(x) \]

or which is the same thing (write \( t = x+h \))

\[ \lim_{h \to 0} \left( \sin(x+h) \right) = \sin(x) \quad \text{[\( \star \)]} \]

Left side of \( \star \) = \( \lim_{h \to 0} \left( \sin(x) \cos(h) + \cos(x) \sin(h) \right) \)

\[ = \sin(x) \lim_{h \to 0} \cos(h) + \cos(x) \lim_{h \to 0} \sin(h) \quad \text{[Limit Laws,} \quad 1 \quad \text{proven in class,} \quad 0 \quad \text{proven (geometrically) in class]} \]

\[ = \sin(x) \cdot 1 + \cos(x) \cdot 0 \]

\[ = \sin(x) \quad & \text{we're done!} \]