Q1. [10 points] Compute the derivatives of the following functions. Show all your work, writing down the names of key differentiation rules you are using. You do not have to spend time simplifying the expressions in your answers.

\[
\frac{d}{dx} \left( (x^3 - 2x^2 + 17) \cos(x) \right) = \frac{d}{dx} \left( x^3 - 2x^2 + 17 \right) \cos(x) + \left( x^3 - 2x^2 + 17 \right) \frac{d}{dx} \cos(x)
\]

\[
= (3x^2 - 2x) \cos(x) + (x^3 - 2x^2 + 17)(-\sin(x))
\]

\[
x^2 - \frac{3}{x} + \frac{7}{\sqrt{x}} - 13
\]

\[
\text{Power (\& Sum) Rule}
\]

\[
\frac{d}{dx} \left( x^2 - \frac{3}{x} + \frac{7}{\sqrt{x}} - 13 \right) = 2x - 3(-1)x^{-2} + 7 \left( \frac{1}{2} \right)x^{-3/2}
\]

\[
= 2x + 3x^{-2} - \frac{7}{2}x^{-3/2}
\]

\[
\frac{2x^2 + 7}{2 \tan(x) + 1}
\]

\[
\text{Quotient Rule}
\]

\[
\frac{d}{dx} \left( \frac{2x^2 + 7}{2 \tan(x) + 1} \right) = \frac{d}{dx} \left( 2x^2 + 7 \right) \frac{1}{2 \tan(x) + 1} - \left( 2x^2 + 7 \right) \frac{d}{dx} \frac{1}{2 \tan(x) + 1}
\]

\[
= 4x \left( 2 \tan(x) + 1 \right) - \left( 2x^2 + 7 \right) \frac{2 \sec^2(x)}{(2 \tan(x) + 1)^2}
\]
Q2]...[10 points] Find $\frac{dy}{dx}$ for each of the following functions. Remember to show the details of your work.

$\sin^2(3x^2 + 9)$

Let $u = \sin(3x^2 + 9)$  
Let $v = 3x^2 + 9$

$y = u^2$  
$u = \sin(v)$

$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dv} \frac{dv}{dx}$  
-- Chain Rule (X2)

$= \frac{du^2}{du} \cdot \frac{d\sin(v)}{dv} \cdot \frac{d(3x^2 + 9)}{dx}$

$= 2u \cdot \cos(v) \cdot 6x$

$= 2\sin(3x^2 + 9) \cdot \cos(3x^2 + 9) \cdot 6x$

$\sqrt{\sec(\tan(x))}$

Let $u = \sec(\tan(x))$

Let $v = \tan(x)$

$y = \sqrt{u} = u^{\frac{1}{2}}$

$u = \sec(v)$

$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dv} \frac{dv}{dx}$  
-- Ch. Rule (X2)

$= \frac{d\sqrt{u}}{du} \cdot \frac{d\sec(v)}{dv} \cdot \frac{d\tan(x)}{dx}$

$= \frac{1}{2} u^{-\frac{1}{2}} \cdot \sec(v) \tan(v) \cdot \sec^2(x)$

$= \frac{\sec(\tan(x)) \cdot \tan(\tan(x)) \cdot \sec^2(x)}{2 \sqrt{\sec(\tan(x))}}$

* A lot of people were confused at this point. Look over this step carefully.
Q3]... [10 points] Find the following limits. Show all your work, carefully stating any important limits from class that you may be using. You can not use theorems (eg. L'Hôpital's Rule) that we have not yet seen in class!

\[
\lim_{t \to 0} \frac{\tan^2(5t)}{t^2} = \lim_{t \to 0} \left( \frac{\sin(5t)}{t} \cdot \frac{1}{\cos(5t)} \right)^2
\]

\[
= \lim_{t \to 0} \left( \frac{\sin(5t)}{5t} \cdot \frac{5}{\cos(5t)} \right)^2
\]

\[
= \left[ \lim_{t \to 0} \left( \frac{\sin(5t)}{5t} \right) \right]^2 \cdot \frac{5^2}{\left[ \lim_{t \to 0} (\cos(5t)) \right]^2}
\]

\[
= 1^2 \cdot \frac{5^2}{1^2}
\]

\[
= 25
\]

\[
\lim_{x \to 1} \frac{\sin(x-1)}{x^2-1} = \lim_{x \to 1} \left( \frac{\sin(x-1)}{(x-1)} \cdot \frac{1}{(x+1)} \right)
\]

\[
= \lim_{x \to 1} \left( \frac{\sin(x-1)}{(x-1)} \right) \cdot \frac{1}{\lim_{x \to 1} (x+1)}
\]

\[
= 1 \cdot \frac{1}{2}
\]

\[
= \frac{1}{2}
\]
Q4. [10 points] Write down the limit of a difference quotient definition of the derivative of a function $f(x)$.

$$f'(x) = \lim_{h \to 0} \left( \frac{f(x+h) - f(x)}{h} \right)$$

State the product rule for differentiation (formula only is enough).

$$\frac{d}{dx} (fg) = \frac{df}{dx} g + f \frac{dg}{dx}$$

Give a proof of the product rule.

$$\frac{d(fg)}{dx} = \lim_{h \to 0} \left( \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \right)$$

$$= \lim_{h \to 0} \left[ \frac{f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)}{h} \right]$$

$$= \lim_{h \to 0} \left[ \left( \frac{f(x+h) - f(x)}{h} \right) g(x+h) + f(x) \left( \frac{g(x+h) - g(x)}{h} \right) \right]$$

$$= \lim_{h \to 0} \left( \frac{f(x+h) - f(x)}{h} \right) \lim_{h \to 0} \left( g(x+h) \right) + f(x) \lim_{h \to 0} \left( \frac{g(x+h) - g(x)}{h} \right)$$

(by continuity of $g(x)$

(follows since $g(x)$ is differentiable)

(since all 3 limits above exist!)

$$= \frac{df}{dx} g(x) + f(x) \frac{dg}{dx}$$

done!
Q5]...[10 points] Find the derivative $y'$ of the implicitly defined curve.

$$y^2 = 2 + x^2$$

$$\Rightarrow 2y \cdot y' = 0 + 2x$$

$$\Rightarrow y' = \frac{2x}{2y} \Rightarrow y' = \frac{x}{y}$$

Find the equation of the straight lines which pass through the point $(0, 1)$ and are tangent to the curve $y^2 = 2 + x^2$ above. How many such tangent lines are there?

But $(x, y)$ on $y^2 = 2 + x^2$ is so that tangent line through it also contains the point $(0, 1)$ (which does not lie on $y^2 = 2 + x^2$).

Slope $= y' = \frac{x}{y}$ from above

$$\Rightarrow \frac{y-1}{x-0} = \frac{x}{y}$$

$$\Rightarrow y^2 - y = x^2$$

But $y^2 = 2 + x^2$ --- point $(x, y)$ lies on the curve $y^2 = 2 + x^2$

Thus $2 + x^2 - y = x^2$

$$\Rightarrow 2 = y$$

$(\sqrt{2}, 2)$ and $(-\sqrt{2}, 2)$ Two points $\implies 2$ tangent lines

$$y - 2 = \frac{\sqrt{2}}{2}(x - \sqrt{2})$$ --- one tangent line

$$y - 2 = \frac{-\sqrt{2}}{2}(x + \sqrt{2})$$ --- 2nd tangent line