Topology I [5853-001]

Hint for Q21 on Page 102.

Let $^-$ denote the closure operation, and ^{c} denote the complement operation. You have to prove the Kuratowski closure-complement result: that there is at most 14 distinct sets that one can produce form a given set $A \subset X$ by applying $^-$ and ^{c}, and that all 14 can be achieved for A a suitable subset of $\mathbb{R}$.

It's easiest to think of the repeated operations as "strings" of $^-$ and ^{c} operations which we apply to a given set A. We'll study the "algebra" of these strings, often writing $string_1 \subset string_2$ instead of the precise expression $A^{string_1} \subset A^{string_2}$. Denote the empty string by 1, so that $A^1 = A$. Proceed as follows.

Step 1. Show that $cc = 1$. That is $(A^c)^c = A$.

Step 2. Show that $^{-\,-} = ^-$. That is $(A^-)^- = A^-$.

Step 3. Show that $(c - c)^n = (c - c)$, for any $n \in \mathbb{Z}_+$.

Step 4. Show that $c - c \subset -$. That is $((A^c)^-)^c \subset A^-$.

Step 5. Note that 1 and 2 imply that we need only look at strings with no two $^-$ operations adjacent (or with no two ^{c} operations adjacent). We still can have $A^{-c-c-c-c-c-c-c-c-}$ for example. Steps 3 and 4 gave us some elementary properties of the operations.

In this step we make a dramatic simplification; namely, we prove that

$$(-c-c)(-c-c) \;=\; (-c-c)$$

Sketch

$$A^{-c-c} \;=\; A^{-(c-c)} \;=\; A^{-(c-c)(c-c)(c-c)} \;\subset\; A^{-(c-c)-(c-c)} \;\subset\; A^{---(c-c)} \;=\; A^{-(c-c)}$$

Justify all the equalities and inclusions along the way. Since the start and end are the same set, we can conclude that all the inclusions are in fact equalities and we're done!

Step 6. Conclude also that

$$-c-c-c- \;=\; -c-$$

Step 7. Now write down all the strings that can possibly give rise to a different set. Keep in mind the results of steps 1–6.

Step 8. Finally, check your 14 candidates above on the set

$$A \;=\; \{1\} \cup \{2\} \cup \mathbb{I}(2,3) \cup [3,4) \cup (4,\infty)$$

in $\mathbb{R}$ with the usual topology. Here $\mathbb{I}(a,b)$ denotes the set of all irrational numbers in the interval (a,b).