MATH 1823 Honors Calculus I
Permutations, Selections, the Binomial Theorem

Permutations. The number of ways of arranging (permuting) n objects is denoted by $n!$ and is called n *factorial*. In forming a particular arrangement (or permutation) we have n choices for the first object. Given a particular choice for the first object, we have $(n-1)$ choices for the second, and so on. Eventually we have $n(n-1)(n-2)\dots(3)(2)(1)$ possible choices in forming an arrangement. Thus there are $n(n-1)(n-2)\dots(3)(2)(1)$ possible arrangements of n objects.

$$n! = n(n-1)(n-2)\dots(3)(2)(1)$$

Note that $1! = 1$. It is an accepted convention to define $0! = 1$.

For example, there are $2! = 2$ arrangements of the letters a, b. Here they are: ab, and ba. There are $3! = 6$ arrangements of the three letters a, b, c. They are: abc, acb, bac, bca, cab and cba.

Q1]... Write out all the arrangements of the 4 letters a, b, c, and d. How many are there?

There are $4! = 24$ of them. Here they are:
abcd, abdc, acbd, acdb, adbc, adcb, bacd, badc, bcad, bcda, bdac, bdca, cabd, cadb, cbad, cbda, cdab, cdba, dabc, dacb, dbac, dbca, dcab, dcba

Q2]... Write down the following numbers (use a calculator to help you) $5!$, $6!$, $7!$, $8!$, $9!$ and $10!$.

$5! = 120$, $6! = 720$, $7! = 5040$, $8! = 40320$, $9! = 362880$, $10! = 3628800$.

As you have guessed from the last exercise above, the factorials grow extremely rapidly. There is a pretty result, due to a dead guy called Stirling, which says that

$$n! \qquad \text{grows like} \qquad \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

This remarkable growth has surprising practical implications. Some of these implications are the reason why some companies invest huge financial resources in developing **fast** algorithms which give **reasonably accurate** results rather than focusing on algorithms which give the best possible results, but at huge costs in computing time.

Take the *traveling salesman problem* for instance. The problem here is one of optimization (actually minimizing travel costs). Suppose a company has to send a salesman to visit n different cities in the mid-western and western U.S. Which order should the salesman visit the cities in order to minimize the total travel costs/time? The answer seems to be very simple: you are told by a travel agent the costs of traveling between all possible pairs of cities. Now you simply *list all possible routes* and compute their total round-trip costs, and then select the round-trips with the lowest costs. This clearly gives the best possible results, but what about the computing time? You think about this for a second: there are n choices for the first city, then $(n-1)$ choices for the second, etc.

Q3]... How many total round trips are there?

There are a total of $n!$ round trips.

So you decide to program a computer to list all the possible round-trips, and to do all the additions quickly to get totals for each round trip. Suppose the computer can compute 1000 round trip totals per second.

Q4]... How long will it take it to deal with 20 cities?
How long will it take it to deal with 25 cities?

It will take 20!/(1000)(60)(60)(24)(365.25) years (the 365.25 term accounts for leap years) to deal with 20 cities. This is a total of 77,094,012.48 years.

It will take 25!/(1000)(60)(60)(24)(365.25) years to deal with 25 cities. This is a total of 49,152,058,594,920 years, or about the time it takes to determine the outcome a presidential election!

Selections. The number of ways of selecting (or choosing) a group of r objects (for example a committee) from a group of n objects is denoted by $\binom{n}{r}$ which is read n *choose* r.

How do we find a formula for $\binom{n}{r}$? Well, first look at all the possible *ordered lists* of r objects that can be selected from n objects. There are n choices for the first member of the list. Given that choice, there are now $(n-1)$ choices for the second member of the list, and so on. Thus there is a total of

$$n(n-1)\dots(n-r+1)$$

possible ordered lists of r things that can be selected from a group of n things. Now, these lists can be combined into groups of size $r!$, where all the lists in a particular group are just arrangements of a given selection of r things from the original group of n. Thus, the total number of choices of r things from the original group of n things is given by

$$\frac{n(n-1)\dots(n-r+1)}{r!}$$

We can tidy this up by multiplying above and below by $(n-r)\dots(2)(1)$ to get

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

We take this as a definition of $\binom{n}{r}$ even if $r = 0$. We remember to use the convention that $0! = 1$.

Q5]... Do this analysis explicitly to determine the number of selections of 3 things from the group of five letters a, b, c, d and e. That is, write down all the ordered lists of size three which can be obtained from these 5 letters (no repetitions within a given list of course!). Then group your lists together if they involve the same three letters. How big is each group? How many groups of lists do you have?

There are 10 groups each containing 6 lists. This is a total of $60 = (5)(4)(3)$. Here they are

abc	*abd*	*abe*	*acd*	*ace*	*ade*	*bcd*	*bce*	*bde*	*cde*
acb	*adb*	*aeb*	*adc*	*aec*	*aed*	*bdc*	*bec*	*bed*	*ced*
bac	*bad*	*bae*	*cad*	*cae*	*dae*	*cbd*	*cbe*	*deb*	*dec*
bca	*bda*	*bea*	*cda*	*cea*	*dea*	*cdb*	*ceb*	*dbe*	*dce*
cab	*dab*	*eab*	*dac*	*eac*	*ead*	*dbc*	*ebc*	*ebd*	*ecd*
cba	*dba*	*eba*	*dca*	*eca*	*eda*	*dcb*	*ecb*	*edb*	*edc*

Q6]... Compute the following numbers: $\binom{0}{0}$, $\binom{1}{0}$, $\binom{1}{1}$, $\binom{2}{0}$, $\binom{2}{1}$, $\binom{2}{2}$, $\binom{3}{0}$, $\binom{3}{1}$, $\binom{3}{2}$, $\binom{3}{3}$, $\binom{4}{0}$, $\binom{4}{1}$, $\binom{4}{2}$, $\binom{4}{3}$, $\binom{4}{4}$. Put your answers down in rows, labeled by the upper number 0, 1, 2, 3, and 4. Do you recognize the result? What is it?

The result is Pascal's triangle. Here it is.

$$\begin{array}{ccccccccc} &&&&\binom{0}{0}&&&& \\ &&&\binom{1}{0}&&\binom{1}{1}&&& \\ &&\binom{2}{0}&&\binom{2}{1}&&\binom{2}{2}&& \\ &\binom{3}{0}&&\binom{3}{1}&&\binom{3}{2}&&\binom{3}{3}& \\ \binom{4}{0}&&\binom{4}{1}&&\binom{4}{2}&&\binom{4}{3}&&\binom{4}{4} \end{array} = \begin{array}{ccccccccc} &&&&1&&&& \\ &&&1&&1&&& \\ &&1&&2&&1&& \\ &1&&3&&3&&1& \\ 1&&4&&6&&4&&1 \end{array}$$

Q7]... Prove that $\binom{n}{r} = \binom{n}{n-r}$. Give an intuitive interpretation of this fact.

We have

$$\binom{n}{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{(n-r)!r!} = \binom{n}{r}$$

and we're done! This result should be intuitively true, since every time you make a selection of r things from a group of n things, you automatically make a selection of $n-r$ things (the remaining or complementary things). Distinct selections have distinct complements. Thus, the number of ways of selecting r things is the same as the number of ways of their $n-r$ complements.

Q8]... Show that $\binom{n}{0}$ and $\binom{n}{n}$ are always equal to 1. Now, we will get the rows of Pascal's triangle, provided we can prove that the $\binom{n}{r}$ add together to give other $\binom{n}{r}$ just like in Pascal's triangle. Prove that

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

Finally, **Q7]** above now confirms our observation of the symmetry in the rows of Pascal's triangle.

We have $\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1$ and so $\binom{n}{n} = \binom{n}{n-n} = \binom{n}{0} = 1$ too.

Now for the addition formula

$$\begin{aligned}
\binom{n}{r} + \binom{n}{r+1} &= \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-(r+1))!} \\
&= \frac{n!}{r!(n-r)!} + \frac{n!}{(r+1)!(n-r-1)!} \\
&= \frac{n!(r+1)}{r!(r+1)(n-r)!} + \frac{n!(n-r)}{(r+1)!(n-r-1)!(n-r)} \\
&= \frac{n!(r+1)}{(r+1)!(n-r)!} + \frac{n!(n-r)}{(r+1)!(n-r)!} \\
&= \frac{n!(r+1) + n!(n-r)}{(r+1)!(n-r)!} \\
&= \frac{n!(r+1+n-r)}{(r+1)!(n-r)!} \\
&= \frac{n!(n+1)}{(r+1)!(n-r)!} \\
&= \frac{(n+1)!}{(r+1)!((n+1)-(r+1))!} \\
&= \binom{n+1}{r+1}
\end{aligned}$$

This also has an intuitive interpretation. Here it is. Suppose you want to select a committee of $r+1$ people from a roomful of $n+1$ people. We know that the total number of possible committees is $\binom{n+1}{r+1}$.

You might like to know how many of those committees contain a particular person (let's call him Paddy!) in the room. Well we can create a committee of $r+1$ people which contains Paddy, by simply choosing Paddy, and then choosing r other people from the remaining n people in the room. There is a total of $\binom{n}{r}$ ways of doing this. Thus there are $\binom{n}{r}$ committees of $r+1$ people which contain Paddy.

What about the Paddy-free committees. Well you simply create these by telling Paddy to leave the room, and then choosing the full committee of $r+1$ from the remaining n people. There are obviously $\binom{n}{r+1}$ ways of doing this. Thus there are $\binom{n}{r+1}$ Paddy-free committees of $r+1$ people.

Now a given committee either contains Paddy or is Paddy-free. Thus, $\binom{n+1}{r+1}$ must be the sum of $\binom{n}{r}$ and $\binom{n}{r+1}$. Done!

Binomial Theorem. This theorem tells you that the $\binom{n}{r}$ are precisely the coefficients of $a^r b^{n-r}$ in the expansion of $(a+b)^n$. Using the summation notation developed in class (ask me if you missed this!) it says

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}$$

We wont give a boring *proof* as is the usual case at this stage, but instead will focus on an intuitive understanding.

Our expression consists of a product of n bracketed terms as shown:

$$(a+b)(a+b)(a+b)\cdots(a+b)$$

Note that the term a^n appears by taking an a out of each bracketed term and multiplying them together. We get a ab^{n-1} by taking an a out of the first bracketed term and b's out of all the remaining bracketed terms. We get another ab^{n-1} by taking the a from the second bracketed term and b's from the others: it appears as $bab \dots b$.

Q9]... How many terms (product of length n consisting of a's and b's in some order) are there altogether?
How many of these give rise to an ab^{n-1}? List these explicitly.

There are 2^n binary words (that's *strings* to you computer science majors!) of length n. We see this by noting that to create such a word, we have two choices for the first letter, two for the second, and so on.

Of these exactly $\binom{n}{1} = n$ give rise to the term ab^{n-1}. Here they are written out explicitly:

$$\begin{array}{c} abbb\dots bb \\ babb\dots bb \\ bbab\dots bb \\ \vdots \\ bbb\dots ba \end{array}$$

The $\binom{n}{1}$ comes from the fact that we have to choose one slot from n in which to place the a and fill the remaining slots with b's.

Q10]... How many terms give rise to an $a^r b^{n-r}$? Remember, we have to *choose* r bracketed expressions from among the list of n bracketed expressions from which to take a's, and then we take b's from the remaining $(n-r)$ bracketed expressions. Hmmmmm....this *choosing* reminds me of something....

We have to choose r places from the n possible positions (in a word of length n) in which to put the a's and fill the remainder with b's. There are $\binom{n}{r}$ ways of doing this. Thus, the coefficient of $a^r b^{n-r}$ is $\binom{n}{r}$. This can also be written as $\binom{n}{n-r}$ if we prefer.

Q11]... Prove that the sum of all the entries in the n-th row of Pascal's triangle is 2^n. Hint, let $a = b = 1$ in the Binomial theorem.

The Binomial Theorem says

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}$$

Setting $a = 1 = b$ gives

$$(1+1)^n = \sum_{j=0}^{n} \binom{n}{j} 1^j 1^{n-j}$$

or

$$2^n = \sum_{j=0}^{n} \binom{n}{j} = \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n}$$