

MATH 1823 Honors Calculus I Irrational Numbers

Due in class on Friday, Sept 8, 2000

Classical belief. The Pythagorean school believed that you could obtain any number (measurement) you like by taking the ratio of two whole numbers (integers). That is, they believed that all numbers were rational. On one level, this might seem to be a reasonable belief. You see you cant make all possible measurements using whole numbers (integers), since the gap between successive whole numbers is 1. We can introduce the fractions $n/2$ to make these gaps all be $1/2$. In general, if we have two fractions, no matter how close together, then their average will be a third fraction whose distance from the first two will be exactly half the distance that they were apart.

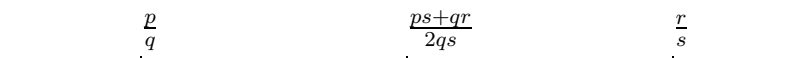
- Show that the average of p/q and r/s is another rational number.

Note that

$$\frac{p/q + r/s}{2} = \frac{ps + qr}{2qs}$$

and that $(ps + qr)$ and $2qs$ are whole numbers with $2qs \neq 0$.

- Draw a picture of these three numbers, starting from the assumption that $p/q < r/s$.



Therefore, since the gap length can be made to go to zero, there must be no gaps at all between the rational numbers. That is, every measurement (number) we want is rational.

Some problem numbers. Here you see that there are some numbers which are not rational.

1. Show that $\sqrt{2}$ (the length of the diagonal of a square of side length 1) is not a rational number.

First we prove a neat fact.

Fact 1. If the square of a whole number is even, then the whole number must also be even.

Proof. Let n be a whole number such that n^2 is even. We have to rule out the fact that n is odd. If n were odd, then we could write $n = 2m + 1$ for some whole number m . Thus, $n^2 = (2m + 1)^2 = 4m^2 + 4m + 1 = 2(2m^2 + 2m) + 1$ which is clearly odd. Thus n cannot be odd.

Now we are ready to argue that $\sqrt{2}$ cannot be rational. We argue by contradiction. Suppose that $\sqrt{2}$ is rational. This means that we can write

$$\sqrt{2} = p/q$$

where the fraction p/q of whole numbers is in its lowest terms. That is, p and q have no factors in common. In particular, this means that we cannot have both p and q being even.

Now square both sides of the equation above to get $2 = p^2/q^2$ and so $p^2 = 2q^2$. Now the right hand side of this last equation is clearly even. Thus the left hand side, namely p^2 , is also even. But p is a whole number, and so Fact 1 tells us that p must be even.

Write $p = 2x$ for some whole number x . Substituting into $p^2 = 2q^2$ yields $4x^2 = 2q^2$ and hence $2x^2 = q^2$. Now the left side of this last equation is clearly even, and therefore the right

side must also be even. That is q^2 is even. But q is a whole number, and so Fact 1 again tells us that q must be even. But if p and q are both even, they must have a factor of 2 in common. This gives our final contradiction, and so the initial assumption of the rationality of $\sqrt{2}$ is false.

2. Show that $\frac{5\sqrt{2}}{24}$ is not rational.

Suppose that there are whole numbers p and q such that $p/q = \frac{5\sqrt{2}}{24}$. This means that $24p/5q = \sqrt{2}$, and so we have that $\sqrt{2}$ is also rational. But this contradicts part 1 above, so $\frac{5\sqrt{2}}{24}$ cannot be rational.

Prove that there are infinitely many irrational numbers.

There are infinitely many rational numbers p/q . The argument used for $\frac{5\sqrt{2}}{24}$ above also shows that $\frac{p\sqrt{2}}{q}$ is irrational for each p/q . Thus we have infinitely many irrationals of the form $\frac{p\sqrt{2}}{q}$.

3. How might you prove that $\sqrt[3]{2}$ is not a rational number?

You would give an (almost verbatim; some equations will have 8's instead of 4's, and squares will be replaced by cubes) version of the proof of the irrationality of $\sqrt{2}$ in 1 above, except that you would appeal to Fact 2 instead of Fact 1.

Fact 2. If the cube of a whole number is even, then the whole number must also be even.

Proof. As in the proof of Fact 1, if $n = 2m + 1$ is an odd whole number (where m is another whole number), then

$$n^3 = (2m + 1)^3 = 8m^3 + 12m^2 + 6m + 1 = 2(4m^3 + 6m^2 + 3m) + 1$$

is also an odd number. Thus the only way that n^3 can be even is for n to be even.

4. How might you prove that $\sqrt{3}$ is not a rational number?

The proof is very similar to the proof of irrationality of $\sqrt{2}$ in 1 above, but this time we cannot talk about even numbers. Instead we have to look at multiples of three. Fact 3 below is the analog of Fact 1 which will be used in the proof.

Fact 3. Suppose that the square of a whole number is a multiple of 3, then the whole number must also be a multiple of 3.

Proof. Let n be a whole number whose square is a multiple of 3. We show that n also must be a multiple of 3, by ruling out the cases where n is not a multiple of three. There are two cases to consider: either n has a remainder of 1 when divided by 3, or n has a remainder of 2 when divided by 3.

In the first case, we can write $n = 3m + 1$ for some other whole number m . Then

$$n^2 = (3m + 1)^2 = 9m^2 + 6m + 1 = 3(3m^2 + 2m) + 1$$

has a remainder of 1 and so is not a multiple of 3.

In the second case, we can write $n = 3m + 2$ for some other whole number m . Then

$$n^2 = (3m + 2)^2 = 9m^2 + 12m + 4 = 3(3m^2 + 4m + 1) + 1$$

also has a remainder of 1 and so is not a multiple of 3.

In conclusion, the only way n^2 can be a multiple of 3 is for n to have been a multiple of 3. Done!

Now we give the proof of the irrationality of $\sqrt{3}$. As in case 1, we argue by contradiction. Start off by assuming that we can write $\sqrt{3} = p/q$ where the fraction of whole numbers p and q is in its lowest terms. In particular, p and q do not have a factor of 3 in common.

Now squaring gives $3 = p^2/q^2$ or $3q^2 = p^2$. The left side of the last equation is clearly divisible by 3. therefore the right side must also be divisible by 3 (since it's equal to the left side!). This means p^2 is divisible by 3. Fact 3 above tells us that p must be divisible by 3. Thus we can write $p = 3x$ for some whole number x . Substituting into the previous equation gives $3q^2 = (3x)^2$ or $3q^2 = 9x^2$ or $q^2 = 3x^2$. Now the right side of this equation is divisible by 3, and so also must the left side be divisible by 3. Thus q^2 is divisible by 3, and fact 3 above ensures that q is divisible by 3. Thus p and q have a factor of 3 in common. This is a contradiction. So our original assumption (that $\sqrt{3}$ could be expressed as a fraction) must be false. That is, $\sqrt{3}$ is irrational.

5. Do a net-search to find answers to the following questions. Say which search engines you used, and which keywords you used!

- (a) Is π rational?

Answer. No, π is irrational.

- (b) Is e rational?

Answer. No, e is irrational.

- (c) Are there more rational numbers than irrational numbers?

Answer. No. In fact, there are many more irrational numbers than rational numbers. We saw in class that you can “list” all the fractions. If you could list all the irrational numbers too, then you could list all the numbers. But we also saw in class that you cannot “list” all the numbers. Thus it must be impossible to list all the irrational numbers.

Conclusions. Now, something must be flawed with the Pythagorean reasoning at the top of the page. Where is the gap? Can you reconcile the two sections of this page [one: gaps between rationals can be made to go to zero, and two: there exist irrational numbers]?

Just because the gap between numbers tends to zero, does not mean that the numbers fill up everything. We can see this clearly by looking at the fractions of the form $\frac{p}{2^n}$ where p is a whole number and $n \in \{0, 1, 2, 3, \dots\}$. Again, the argument in the first paragraph tells us that the gap distances between such numbers can be made arbitrarily small, but we clearly don't have the number $1/3$. So, even though, we have a collection of numbers with arbitrarily small gap distances, we don't have everything. The same thing can happen (and does!) with the rational numbers.