Archive of Calculus IV Questions

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This is an archive of past Calculus IV exam questions. You should first attempt the questions without looking at any hints or solutions. If you get stuck on a particular question, then you should first read the hints. You should only need to look at the solutions as a last resort, or to confirm your own solution.

If you prefer to look at past exams, you can visit my teaching page or you can visit the Calculus at OU resources page.

1 The Questions

1.1 Path Integrals, Green's Theorem

Q1. Let $\mathbf{F} = \langle 2zx + \sin y, x\cos y, x^2\rangle$. Does there exist a function $\phi(x, y, z)$ such that $\nabla\phi = \mathbf{F}$? If so, find such a function ϕ.

Let $\mathbf{F}$ be the conservative vector field defined above in the first part of this question. Determine the value of the path integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ where C is the path given by $\mathbf{r}(t) = \langle \sin t, t, \cos t\rangle$ for $0 \le t \le 3\pi/4$.

Want a hint? Go to the answer.

Q2. Write down the equation that appears in Green's theorem, stating what each part means.

Want a hint? Go to the answer.

Q3. Use Green's Theorem to show that if C is a smooth, positively oriented, simple, closed curve then $\frac{1}{2}\int_C xdy - ydx$ represents the area of the region enclosed by C.

Use a line integral to determine the area of the planar region enclosed by the curve

$$\mathbf{r}(t) \;=\; \langle a\cos^3 t,\, b\sin^3 t\rangle \qquad 0 \le t \le 2\pi\,.$$

Want a hint? Go to the answer.

Q4. Use Green's theorem to evaluate the path integral of the vector field $\mathbf{F} = \langle xy, y^5\rangle$ around the positively oriented triangle C with vertices $(0, 0)$, $(2, 0)$, and $(2, 1)$.

Want a hint? Go to the answer.

1.2 Parametric Surfaces, Surface Integrals

Q1. You have just been hired by the POCKETGOPHER mining and tunneling company to help them design their next generation drill bit. The basic drill design consists of a cylindrical axis with a helical blade as shown in the diagram. Now the people at POCKETGOPHER know that the friction on the drill bit depends on surface area, and they know the surface area of a cylinder, but they need your calculus expertise when it comes to helical surfaces.

Give a parametric description of the helical surface with helix boundary curves given by $\langle \cos t, \sin t, t \rangle$ for $0 \leq t \leq 2\pi$ and by $\langle 2\cos t, 2\sin t, t \rangle$ for $0 \leq t \leq 2\pi$.

Want a hint? Go to the answer.

Compute the surface area of the helical surface that you parameterized above.

Want a hint? Go to the answer.

1.3 Change of variables

Q1. Write down the change of variables formula for triple integrals.

Want a hint? Go to the answer.

Q2. Use the change of variables formula to evaluate the volume of the ellipsoid bounded by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Want a hint? Go to the answer.

1.4 Cylindrical and Spherical coordinates

Q1. Write down the expression for volume element dV in spherical coordinates.

Want a hint? Go to the answer.

Q2. Use spherical coordinates to compute the volume of the solid which lies below the sphere $x^2+y^2+z^2 = 9$ and above the (upper half ($z > 0$) of the) cone $z^2 = 3(x^2 + y^2)$.

Want a hint? Go to the answer.

Q3. Write the following triple integral out as a spherical coordinates triple integral.

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{\sqrt{9-x^2-y^2}} z(x^2+y^2+z^2)dz$$

Want a hint? Go to the answer.

Q4. Sketch the region which is described in the following triple integral.

$$\int_{0}^{\pi/4}\int_{0}^{\pi/2}\int_{0}^{\sec\phi} \rho^2 \sin\phi d\rho d\theta d\phi$$

Want a hint? Go to the answer.

Q5. Evaluate the following triple integral by first sketching the region of integration, and then converting it to a spherical coordinates integral.

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z\, dzdxdy$$

Want a hint? Go to the answer.

2 The Hints

2.1 Path Integrals, Green's Theorem

H1. If the curl of $\mathbf{F}$ is zero, then $\mathbf{F}$ is conservative. We can then find ϕ by solving the equation $\nabla\phi = \mathbf{F}$.

By the first part of this question we know that $\mathbf{F} = \nabla\phi$. Therefore, the path integral becomes $\int_C \nabla\phi \cdot d\mathbf{r}$. There is a theorem which enables us to evaluate such path integrals very easily.

Back to the question. Go to the answer.

H2. This relates closed path integrals with area integrals in the plane.

Back to the question. Go to the answer.

H3. Use the statement of Green's Theorem. Basically P and Q are chosen for you.

Just use the first part and evaluate the path integral. Trig identities will ensure that the resulting definitel integral will be fairly straightforward.

Back to the question. Go to the answer.

H4. This is a direct application of Green's Theorem. Things will simplify down to a very standard double integral.

Back to the question. Go to the answer.

2.2 Parametric Surfaces, Surface Integrals

H1. You should think of this surface as being made up of *horizontal* (parallel to xy-plane) line segments which begin on one helix and end on the other like a spiral staircase with infinitely many (infinitesimally small) steps!

No real hint for second part of question. This is a standard surface area computation using double integrals.

Back to the question. Go to the answer.

2.3 Change of variables

H1. Remember that the image of a small rectangular box in uvw-space can be approximated by a parallelepiped in xyz-space with edge vectors $\langle x_u, y_u, z_u\rangle du$, $\langle x_v, y_v, z_v\rangle dv$, and $\langle x_w, y_w, z_w\rangle dw$. Volumes of parallelepipeds can be computed using triple products (Calculus III).

Back to the question. Go to the answer.

H2. Note that the ellipsoid equation can be written as

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

which looks very like the equation of a unit sphere if we were to use coordinates x/a instead of x etc.

Back to the question. Go to the answer.

2.4 Cylindrical and Spherical coordinates

H1. You should think intuitively about the dimensions of the almost rectangular box obtained by perturbing ρ, θ and ϕ slightly.

Back to the question. Go to the answer.

H2. This solid has axial symmetry about the z-axis. What does this tell you about the θ limits?

The solid is obtained by coning a patch on the sphere of radius 3 back to the origin. What does this tell you about the ρ limits?

The trickiest limits to determine are those for ϕ. Note that the cone equation reads $z^2 = 3r^2$ or $z = \sqrt{3}r$ since it's a cone with $z \geq 0$. Draw an rz-cross section of the cone. Do you recognize any right angled triangles? What are the ϕ limits?

Don't forget the spherical volume element, when writing down your integral.

Back to the question. Go to the answer.

H3. There are two steps to this question. First you need to determine what the region of integration looks like. You do this by interpreting the limits of integration in the given triple integral. Keep in mind that you should expect spheres/cones/ etc as these translate easily into spherical coordinates. Next you have to write down the spherical coordinates triple integral. This means remembering the spherical volume element, translating the integrand, and describing the region using spherical coordinates (to get limits of integration).

Back to the question. Go to the answer.

H4. Remember that spherical volume elements piece together to give cones which radiate outwards from the origin. What do the ρ limits tell us about these cones. Now look at the ϕ and θ limits.

Back to the question. Go to the answer.

H5. This question is much the same as Question 3, so you can look at that hint.

Back to the question. Go to the answer.

3 The Answers

3.1 Path Integrals, Green's Theorem

A1. Note that the curl of $\mathbf{F}$ is zero.

$$curl\mathbf{F} \;=\; \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2zx + \sin y & x\cos y & x^2 \end{vmatrix} \;=\; \langle\, 0-0,\ 2x-2x,\ \cos y - \cos y \,\rangle \;=\; \langle\, 0,\ 0,\ 0 \,\rangle$$

Since $\mathbf{F}$ is defined on all of $\mathbf{R}^3$, we conclude that $\mathbf{F}$ is conservative.

Now for the second part. We want $\langle \phi_x, \phi_y, \phi_z \rangle = \mathbf{F}$. Thus we must solve the following equations

$$\phi_x \;=\; 2zx + \sin y\,; \qquad \phi_y \;=\; x\cos y\,; \qquad \phi_z \;=\; x^2\,.$$

The first equation gives $\phi = x^2z + x\sin y$ plus some function of y and z. It is easy to see from the second and third equations that this function is a constant, and so we get

$$\phi(x,y,z) \;=\; x^2z \,+\, x\sin y \,+\, C\,.$$

By the fundamental Theorem of Path Integrals we have

$$\begin{aligned} \int_C \mathbf{F}\cdot d\mathbf{r} \;=\; \int_C \nabla\phi\cdot d\mathbf{r} \;&=\; \phi(\mathbf{r}(3\pi/4)) \,-\, \phi(\mathbf{r}(0)) \\ &=\; \phi(\sqrt{2}/2, 3\pi/4, -\sqrt{2}/2) \,-\, \phi(0,0,1) \\ &=\; -\sqrt{2}/4 + 1/2\,. \end{aligned}$$

Back to the question. Back to the hint.

A2. Green's Theorem states that

$$\int_C P\,dx \,+\, Q\,dy \;=\; \iint_R [Q_x - P_y]\; dA$$

where C is a smooth, positively oriented, simple, closed curve which bounds a region R in the plane, and P and Q have continuous partial derivatives on R.

Back to the question. Back to the hint.

A3. The first part is an immediate application of Green's Theorem

$$\frac{1}{2}\int_C x\,dy \,-\, y\,dx \;=\; \frac{1}{2}\iint_R \left[\frac{\partial x}{\partial x} \,-\, \frac{\partial(-y)}{\partial y}\right] dA \;=\; \frac{1}{2}\iint_R 2\,dA \;=\; \iint_R dA$$

which is precisely the area enclosed by the curve C.

For the second part we just convert the area into a path integral as described in part one.

$$
\begin{aligned}
\text{Area} &= \frac{1}{2}\int_C x\,dy \;-\; y\,dx \\
&= \frac{1}{2}\int_0^{2\pi} (a\cos^3 t)(3b\sin^2 t\cos t) - (b\cos^3 t)(3\cos^2 t)(-\sin t)\,dt \\
&= \frac{3ab}{2}\int_0^{2\pi} \cos^2 t\sin^2 t(\cos^2 t + \sin^2 t)\,dt \\
&= \frac{3ab}{2}\int_0^{2\pi} \cos^2 t\sin^2 t\,dt \\
&= \frac{3ab}{2}\int_0^{2\pi} \left(\frac{1+\cos 2t}{2}\right)\left(\frac{1-\cos 2t}{2}\right)\,dt \\
&= \frac{3ab}{8}\int_0^{2\pi} 1-\cos^2(2t)\,dt \\
&= \frac{3ab}{8}\int_0^{2\pi} \frac{1-cos4t}{2}\,dt \\
&= \frac{3\pi ab}{8}
\end{aligned}
$$

Back to the question. Back to the hint.

A4. Let T denote the triangular region enclosed by C. Then Green's Theorem states that

$$
\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_T \left[\frac{\partial(y^5)}{\partial x} \;-\; \frac{\partial(xy)}{\partial y}\right] dA \\
&= \iint_T -x\,dA \\
&= -\int_0^1\int_{2y}^2 x\,dx\,dy \\
&= -\int_0^1 \left[x^2/2\right]_{2y}^2\,dy \\
&= \int_0^1 [2y^2-2]\,dy \\
&= \left[2y^3/3-2y\right]_0^1 \\
&= -4/3
\end{aligned}
$$

Back to the question. Back to the hint.

3.2 Parametric Surfaces, Surface Integrals

A1. The hint tells us to consider the surface as being made up of infinitely many horizontal line segments with one endpoint on the first helix and the other endpoint on the second helix. So the horizontal line segment at height t begins at the point $(\cos t, \sin t, t)$ and ends at the point $(2\cos t, 2\sin t, t)$. Thus points on this line segment may be parameterized as follows

$$\langle 0, 0, t\rangle \; + \; s\langle \cos t, \sin t, 0\rangle \qquad\qquad \text{for } 1 \le s \le 2.$$

This gives the following parametric description for the helical surface

$$\langle x(s,t), y(s,t), z(s,t)\rangle \;=\; \langle s\cos t, s\sin t, t\rangle \qquad\qquad \text{for } 1 \le s \le 2 \text{ and } 0 \le t \le 2\pi.$$

Recall that surface area is given by the double integral

$$\iint \sqrt{\left(\frac{\partial(x,y)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(y,z)}{\partial(s,t)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(s,t)}\right)^2}\, ds\, dt$$

over the rectangular region $[1,2] \times [0, 2\pi]$ in the st-plane. So we have to compute the three determinants first of all.

Now,

$$\frac{\partial(x,y)}{\partial(s,t)} \;=\; \begin{vmatrix} x_s & x_t \\ y_s & y_t \end{vmatrix} \;=\; \begin{vmatrix} \cos t & -s\sin t \\ \sin t & s\cos t \end{vmatrix} \;=\; s$$

and

$$\frac{\partial(y,z)}{\partial(s,t)} \;=\; \begin{vmatrix} y_s & y_t \\ z_s & z_t \end{vmatrix} \;=\; \begin{vmatrix} \sin t & s\cos t \\ 0 & 1 \end{vmatrix} \;=\; \sin t$$

and

$$\frac{\partial(z,x)}{\partial(s,t)} \;=\; \begin{vmatrix} z_s & z_t \\ x_s & x_t \end{vmatrix} \;=\; \begin{vmatrix} 0 & 1 \\ \cos t & -s\sin t \end{vmatrix} \;=\; -\cos t\,.$$

This gives a surface area integral of

$$\int_0^{2\pi}\int_1^2 \sqrt{s^2+1}\, ds\, dt \;=\; 2\pi \int_1^2 \sqrt{s^2+1}\, ds\,.$$

You would get full marks for this much. You may evaluate this last integral by a trig substitution (or by looking it up in the table) to get a final answer of $2\pi[s/2\sqrt{s^2+1} + 1/2\ln(s + \sqrt{s^2+1})]_1^2$.

Back to the question. Back to the hint.

3.3 Change of variables

A1. Suppose the change of variables $(x(u,v,w), y(u,v,w), z(u,v,w))$ takes a region S in uvw-space to a region R in xyz-space. Then

$$\iiint_R f(x,y,z)\,dV \;=\; \iiint_S f(x(u,v,w), y(u \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| du\,dv\,dw$$

where

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \;=\; \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

Back to the question. Back to the hint.

A2. Recall that

$$\text{Volume} \;=\; \iiint_{\text{ellipsoid}} dV$$

Let $x = au$, $y = bv$, and $z = cw$. Then the ellipsoid above becomes a unit ball bounded by the sphere $u^2 + v^2 + w^2 = 1$ in uvw-space. We also have $x_u = a$, $x_v = x_w = 0$, $y_v = a$, $y_u = y_w = 0$, and $z_w = a$, $z_u = z_v = 0$. Thus we get

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} \;=\; \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \;=\; abc$$

and, substituting into the change of variables formula yields

$$\iiint_{\text{ellipsoid}} dV \;=\; \iiint_{\text{unit ball}} abc\,du\,dv\,dw \;=\; abc(\text{Volume of unit ball}) \;=\; \frac{4\pi}{3}abc\,.$$

Back to the question. Back to the hint.

3.4 Cylindrical and Spherical coordinates

A1. The volume element in Spherical coordinates is given by

$$dV \;=\; \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

Back to the question. Back to the hint.

A2. The equation of the cone is $z^2 = 3r^2$ or $z = \sqrt{3}r$ since we're looking at upper half of cone. The cone angle (between positive z-axis and slant surface of cone) is equal to the upper limit of ϕ and is given by $\tan\phi = r/z = 1/\sqrt{3}$. Thus ϕ has a upper limit of $\pi/6$, and the solid has the following spherical coordinates description

$$0 \le \rho \le 3\,; \qquad 0 \le \theta \le 2\pi\,; \qquad 0 \le \phi \le \pi/6$$

Therefore the volume is given by

$$\begin{aligned}\int_0^{\pi/6}\int_0^{2\pi}\int_0^3 \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi \;&=\; \left[\rho^3/3\right]_0^3 [\theta]_0^{2\pi} [-\cos\phi]_0^{\pi/6} \\ &=\; (9)(2\pi)(1-\sqrt{3}/2) \\ &=\; 18\pi(1-\sqrt{3}/2)\,.\end{aligned}$$

Back to the question. Back to the hint.

A3. The region is precisely one quarter of a solid ball which is centered on the origin and has radius 3. The quarter is is above the xy-plane and to the positive y half of the xz-plane.

The spherical coordinates description of this region is just

$$0 \le \rho \le 3\,, \qquad 0 \le \theta \le \pi\,, \qquad 0 \le \phi \le \pi/2\,.$$

Noting that the integrand converts into $\rho\cos\phi(\rho^2)$, and remembering that $dv = \rho^2 \sin\phi d\rho d\theta d\phi$, we obtain

$$\int_0^{\pi/2}\int_0^{\pi}\int_0^3 \rho^5 \cos\phi \sin\phi d\rho d\theta d\phi\,.$$

Back to the question. Back to the hint.

A4. We build the region up from small blocks which radiate outwards from the origin ($\rho = 0$) until the horizontal plane $z = 1$ ($\rho = \sec\phi$). We see this last fact from the definition of spherical coordinates as follows: $\rho = \sec\phi \;\Rightarrow\; \rho\cos\phi = 1 \;\Rightarrow\; z = 1$. These blocks produce a type of cone with vertex at the origin and base on the plane $z = 1$. Now we rotate these cones one quarter way about the z axis – from the positive x-axis ($\theta = 0$) until the positive y-axis ($\theta = \pi/2$). Finally, we build up copies of this region from the vertical ($\phi = 0$) until an angle of $\pi/4$ with the z-axis ($\phi = \pi/4$).

Our resulting region is one quarter of a solid cone of height 1 and cone angle (between axis and side of cone) of $\pi/4$. The cone has vertex at the origin, and "base" on the pane $z = 1$. The quarter corresponds to the first octant $x \ge 0, y \ge 0, z \ge 0$.

Back to the question. Back to the hint.

A5. The z-coordinates run from the cone $z = \sqrt{x^2 + y^2}$ up to the sphere $x^2 + y^2 + z^2 = 2$. These two surfaces intersect in a unit circle $x^2 + y^2 = 1$ at height $z = 1$. The cone angle is just $\pi/4$. The x and y limits tell us that we're integrating over the the part of this region between the cone and sphere which lies over the left half ($x \le 0$) of the unit disk in the xy-plane.

Thus the region of integration has the following spherical coordinates description

$$0 \le \rho \le \sqrt{2}\,, \qquad \pi/2 \le \theta \le 3\pi/2\,, \qquad 0 \le \phi \le \pi/4\,.$$

The integral becomes

$$\begin{aligned}
\int_0^{\pi/4}\int_{\pi/2}^{3\pi/2}\int_0^{\sqrt{2}} \rho\cos\phi\rho^2\sin\phi\,d\rho\,d\theta\,d\phi &= \left[\frac{\rho^4}{4}\right]_0^{\sqrt{2}} [\theta]_{\pi/2}^{3\pi/2} \left[\frac{\sin^2\phi}{2}\right]_0^{\pi/4} \\
&= (4)(\pi)(1/4) \\
&= \pi\,.
\end{aligned}$$

Back to the question. Back to the hint.

4 Figures

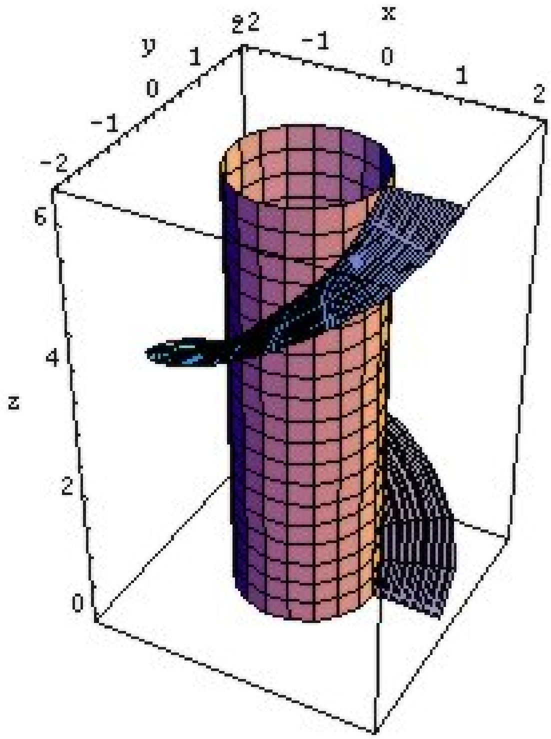

Figure 1: The helical surface and the central cylinder

Back to the question.