BRANCHED COVERINGS OF CUBICAL COMPLEXES AND SUBGROUPS OF HYPERBOLIC GROUPS

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Abstract. By considering branched coverings of piecewise euclidean cubical complexes, we produce an example of a torsion free hyperbolic group containing a finitely presented subgroup which is not hyperbolic.

§1. Introduction

One of the fundamental notions in geometric group theory is that of a hyperbolic group. This concept was introduced in [Gr] as a generalization of both the fundamental group of a closed negatively curved manifold and of finitely generated free groups. Hyperbolic groups are finitely presented, and have solvable word and conjugacy problems. In addition, torsion free hyperbolic groups have finite Eilenberg-Mac Lane spaces, and so they are of finite type.

It is natural to inquire if hyperbolicity is an hereditary property of groups: i.e. to ask if subgroups of hyperbolic groups are themselves hyperbolic. The answer to this is clearly negative, as there are infinitely generated subgroups of the free group on two generators. For example, the kernel of the epimorphism $F_2 \to \mathbb{Z}$ which sends both generators of F_2 to the generator $1 \in \mathbb{Z}$ is not finitely generated.

E. Rips gave the first example of a hyperbolic group which has a finitely generated subgroup which is not finitely presentable, and hence not hyperbolic (see [R]). In [BM] there is an example of a finitely generated but not finitely presentable subgroup of a cocompact 4-dimensional Kleinian group. Now the question becomes: Is every finitely presented subgroup of a hyperbolic group necessarily hyperbolic?

In his fundamental paper [Gr], Gromov gave a sketch of what appeared to be a hyperbolic group G containing a finitely presented subgroup which is not hyperbolic. The group G is the fundamental group of a closed 5-dimensional manifold, realized as a finite branched cover of the 5-torus, branched over generically chosen codimension 2 subtori. The subgroup is the kernel of a homomorphism $G \to \mathbb{Z}$ corresponding to a fibration of the 5-manifold over the circle. If the group G were hyperbolic, one could argue that the

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finitely presented subgroup was not hyperbolic. However, M. Bestvina [Be] has shown that the universal cover of any such branched cover of the 5-torus must contain isometrically embedded flat planes, and hence, that the group G is not hyperbolic.

There are situations where one is guaranteed that a finitely presented subgroup of a hyperbolic group is also hyperbolic. S. Gersten [Ge] showed that if a hyperbolic group has cohomological dimension 2 then every finitely presented subgroup is necessarily hyperbolic.

In this paper we produce a hyperbolic group which contains a finitely presented subgroup which is not hyperbolic. Here is the precise result.

Theorem 1.1. *There exists a short exact sequence of groups*

$$1 \to H \to G \to \mathbb{Z} \to 1$$

such that,

(1) *G is torsion free hyperbolic,*
(2) *H is finitely presented, and*
(3) *H is not of type FP_3.*

In particular, H is not hyperbolic.

The group G in this example has cohomological dimension 3, and hence the bound in Gersten's result is sharp.

The hyperbolic group G is obtained as the fundamental group of a compact, piecewise euclidean cubical complex Y, whose universal cover is a δ-hyperbolic metric space. The δ-hyperbolicity is ensured by taking a suitable branched covering of another cubical complex.

The epimorphism $G \to \mathbb{Z}$ comes from a continuous map $Y \to S^1$. This map lifts to the universal covers to give a "Morse function" $f : X \to \mathbb{R}$. The kernel H acts cocompactly by deck transformations on point preimages or "level sets" of f. The Morse theory viewpoint developed in [BB] yields topological information about the level sets and hence finiteness properties of the kernel H.

This paper is organised as follows. In section 2 we define and describe properties of hyperbolic groups and of CAT(0) metric spaces. In Section 3 we define PE cubical complexes, and describe conditions on such complexes which ensure that their universal covers are CAT(0). We define the notion of a Morse function on a PE cubical complex in Section 4, and show how Morse functions can be used to deduce finiteness properties of certain groups. In Section 5 we define and study the notion of a branched covering of a PE cubical complex. In Section 6 branched coverings are used to construct a specific PE cubical complex together with a Morse function. The main result, Theorem 1.1, follows easily from this construction. Finally, in Section 7 we pose some interesting questions regarding finiteness properties of subgroups of hyperbolic groups.

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§2. Hyperbolic Groups and CAT(0) Metric Spaces

Let G be a finitely generated group with finite generating set $A \subset G$ which is closed with respect to inversion. We define the *Cayley graph* of G with respect to the generating set A (denoted by $\Gamma_A(G)$) as follows. The vertex set of $\Gamma_A(G)$ is the group G, and for each $a \in A$ and each $g \in G$ there is a edge from the vertex g to the vertex ga. We make $\Gamma_A(G)$ into a metric space by requiring that each edge be isometric to the unit interval, and then taking the induced path metric.

A path connected metric space X with distance function d is called a *geodesic* metric space if for all points x, y in X there is an isometric map from the interval $[0, d(x, y)]$ to a path in X joining x to y. For example, if G is a finitely generated group then the Cayley graph of G with the usual path metric is a geodesic metric space. Like the example above, the geodesic metric spaces we consider will be complete and locally compact.

Given three points x, y, z in a geodesic metric space X, choose geodesics joining them to form a triangle Δxyz. Denote the geodesic sides of this triangle by $[x, y]$, $[x, z]$ and $[y, z]$. We say that the geodesic triangle Δxyz is *δ-thin* if the distance of any point on one side to the union of the other two sides is bounded above by δ. That is

$$d(\,p\,, [y, z] \cup [x, z]) \;\leq\; \delta \qquad \text{for all } p \in [x, y]$$

and similarly for any permutation of the sides.

A path connected geodesic metric space X is called *δ-hyperbolic* if there exists (a universal) $\delta \geq 0$ such that every geodesic triangle in X is δ-thin. A finitely generated group G is called *hyperbolic* if the Cayley graph $\Gamma_A(G)$ is a δ-hyperbolic metric space. In [S] it is shown that if B is another finite generating set then $\Gamma_B(G)$ is δ'-hyperbolic for some $\delta' \geq 0$, so the property of being a hyperbolic group is independent of the generating set.

We record some results about hyperbolic groups which will be used later on. The first result gives a useful way of recognizing a hyperbolic group. For a proof see [GdH] page 59.

Proposition 2.1. *Suppose that a group G acts cocompactly, properly, by isometries on a δ-hyperbolic metric space. Then G is a hyperbolic group.* $\square$

The δ-hyperbolic spaces we consider in this paper are CAT(0) metric spaces which have no embedded flat planes. Let X be a metric space. Given three points x, y, z of X let x', y', and z' be the vertices of a *comparison triangle* in the euclidean plane: that is, $d(x, y) = d'(x', y')$, $d(y, z) = d'(y', z')$, and $d(z, x) = d'(z', x')$ where $d(\ ,\ )$ denotes the metric on X and $d'(\ ,\ )$ denotes the euclidean metric. We say that the complete, locally compact, geodesic metric space X satisfies the CAT(0) *comparison inequaltiy* or that X is a CAT(0) *metric space* if for all geodesic triangles xyz in X and points $p \in [y, z]$ we have

$$d(x, p) \;\leq\; d'(x', p')$$

where $x'y'z'$ is a comparison triangle for xyz and $p' \in [y', z']$ is such that $d(p, y) = d'(p', y')$.

There is a nice way of characterizing those CAT(0) metric spaces which are also δ-hyperbolic in terms of embedded flat planes. See [Gr] and [Br].

Theorem 2.2 (Eberlein, Gromov, Bridson). *Let X be a CAT(0) metric space which has a cocompact group of isometries, and which does not contain any isometric embeddings of the euclidean plane. Then X is a δ-hyperbolic metric space.* $\square$

The next result states that torsion free hyperbolic groups are of finite type. It will be used in Section 6 to conclude that the subgroup H in Theorem 1.1 is not hyperbolic. A proof can be found in [Gr] page 101 or [GdH] page 64.

Theorem 2.3 (Rips). *Let H be a hyperbolic group. There exists a locally finite, contractible, finite dimensional simplicial complex (called the Rips complex of H) on which H acts faithfully, properly, simplicially, and cocompactly. In particular, if H is torsion free, then the action is free and the quotient of this complex by H is a finite Eilenberg-Mac Lane space $K(H,1)$.* $\square$

§3 Piecewise euclidean cubical complexes

Our construction uses piecewise euclidean complexes. We will need to know when the universal covers of such complexes become CAT(0) metric spaces in the induced path metric. It turns out that a simple combinatorial condition involving the notion of a flag complex suffices.

Definition 3.1. A simplicial complex L is said to be a *flag* complex if every finite collection of vertices of L which are pairwise adjacent spans a simplex in L. We see that such complexes L are completely determined by their 1-skeleton.

There are many examples of flag complexes. In fact, the first barycentric subdivision of any simplicial complex is a flag complex. We leave the verification of this as an exercise for the reader.

By a *regular n-cube* $\square^n$ we mean a cube which is isometric to the cube in $\mathbb{R}^n$ with vertex at the origin, and edges defined by the unit basis vectors.

Definition 3.3. A *piecewise euclidean cubical complex (of dimension $\leq N$)* is a cell complex X satisfying the following conditions.

• The characteristic map of an n-cell e ($n \leq N$) is a map $\chi_e : \square^n \to e$, where $\square^n$ is a regular n-cube in $\mathbb{R}^N$. A map $f : \square^n \to e$ is said to be an *admissible characteristic map* if it is just χ precomposed with a partial isometry of $\mathbb{R}^N$.

• The restriction of χ_e to any face of $\square^n$ is an admissible characteristic map of a cell of X.

We shall refer to these as PE cubical complexes. Informally, we think of a finite PE cubical complex as being constructed from a finite, disjoint collection of regular cubes, by glueing their faces via isometries.

We now examine the local geometry of a PE cubical complex.

Definition 3.4. Let f be a face of a regular cube $\square^n \subset \mathbb{R}^n$, and let $x \in f$ be the center of this face. In the case f is a vertex, we just let x be this vertex. Define the *link* of f in $\square^n$ (denoted by $Lk(f, \square^n)$) to be the set of all unit tangent vectors at $x \in \mathbb{R}^n$ which are orthogonal to f and which point into $\square^n$. It is a subset of the unit sphere which is homeomorphic to a simplex of dimension $n-(\dim(f)+1)$. It has a natural spherical metric in which all dihedral angles are right angles. Such spherical simplices are termed *all right* simplices.

Thus the link of a cell in a PE cubical complex inherits the satructure of a piecewise spherical cell complex, all of whose cells are all right simplices. The next definition gives a combinatorial version of nonpositive curvature for PE cubical complexes.

Definition 3.5. A PE cubical complex is said to be *nonpositively curved* if the link of each vertex is a flag complex.

The following lemma of Gromov [Gr] (pg. 122) (see also [Da] Lemma 9.3) tells when a piecewise spherical cell complex, whose cells are all right simplices, satisfies the CAT(1) inequality. (Gromov's statement uses the "no triangle condition" instead of the "flag complex" condition below).

Lemma 3.6 (Gromov). *Let L be an all right, piecewise spherical complex. Then L is a CAT(1) space if and only if it is a flag complex.* $\square$

A PE cubical complex whose links are all CAT(1) spaces satisfies a local CAT(0) inequality (see [Gr] pg. 120 or [GH] pg. 197), and so its universal cover is a CAT(0) metric space (and hence is contractible) — see [Gr] pg. 119 or [GH] pg. 193. There is a nice survey of these results in section I of [Da].

§4. Morse Theory and Finiteness Properties of Groups

We will consider the following finiteness conditions for groups. A group H is said to be *of type* F_n if there is a $\mathrm{K}(H,1)$ with finite n-skeleton. A group H is said to be *of type* FP_n if there exists a resolution (exact sequence)

$$P_n \to P_{n-1} \to \cdots \to P_0 \to \mathbb{Z}$$

of the trivial $\mathbb{Z}H$-module $\mathbb{Z}$ by finitely generated projective $\mathbb{Z}H$-modules P_i.

An application of Schanuel's Lemma from homological algebra gives us the following method of showing that a group H is of type FP_n but not of type FP_{n+1}. A proof of this can be found in K. Brown's book [Br 1] (pp. 192-193) for example.

Proposition 4.1. *If there exists a resolution*

$$0 \to Z_n \to P_n \to \cdots \to P_0 \to \mathbb{Z} \to 0$$

where the P_i are all finitely generated, projective over $\mathbb{Z}H$ and Z_n is not finitely generated over $\mathbb{Z}H$. Then H is of type FP_n but not of type FP_{n+1}. $\square$

The following standard result shows how one can use topological information about a group acting on a cell complex to conclude that the group satisfies various finiteness conditions.

Proposition 4.2. *Suppose that the group H acts freely, properly, cellularly, and cocompactly on the cell complex X. Then the following implications hold.*

(1) *If $\widetilde{H}_i(X, \mathbb{Z}) = 0$ for $0 \leq i \leq n-1$ and $\widetilde{H}_n(X, \mathbb{Z})$ is not finitely generated as a $\mathbb{Z}H$-module, then H is of type FP_n but not of type FP_{n+1}.*

(2) *If X is $(n-1)$-connected, then H is of type F_n.* $\square$

As an example, a torsion free hyperbolic group has a finite Eilenberg-Mac Lane space by Theorem 2.3 and so is of finite type. In particular, we see that it is of type FP_n for all integers $n \geq 0$. This is how we show that the torsion free subgroup H in our example is not hyperbolic. If H were hyperbolic it would be of type FP_3, contradicting Theorem 1.1 (3).

As indicated in the introduction, we shall produce groups satisfying certain finiteness conditions by studying *Morse functions* on PE cubical complexes. A general version of Morse theory for affine complexes was introduced in [BB] where it was used to study finiteness properties of certain subgroups of right angled Artin groups. We recall the relevant definitions and results below for the category of PE cubical complexes.

Definition 4.3. Suppose that the group G acts freely, cocompactly, properly, by isometries on a contractible PE cubical complex X. Let $\phi : G \to \mathbb{Z}$ be an epimorphism, and let $\mathbb{Z}$ act on $\mathbb{R}$ by integer translations.

We say that the continuous map $f : X \to \mathbb{R}$ is a *ϕ-equivariant Morse function* on X if it satisfies the following properties.

- $f \circ g \;=\; \phi(g) \circ f$ for all $g \in G$.
- For each n-cell e of X with characteristic map $\chi_e : \square^n \to X$, the composition

$$f\chi_e : \square^n \to \mathbb{R}$$

extends to an affine map $\mathbb{R}^n \to \mathbb{R}$, and $f\chi_e$ is constant only for $n = 0$.

- The f-image of the 0-skeleton of X is discrete in $\mathbb{R}$.

We drop the adjective *ϕ-equivariant* if there are no group actions, and f still satisfies properties two and three.

It is observed in [BB] that PE cubical complexes have a natural PL structure. Thus, given a cell e in a PE complex X with characteristic map $\chi_e : \square_e \to e$, and a vertex $w \in \square_e$ mapping to $v \in e$, we have a well-defined map of links

$$\chi_{e*} : Lk(w, \square_e) \;\to\; Lk(v, e) \;\subset\; Lk(v, X)$$

We can view $Lk(v, X)$ as a union

$$Lk(v, X) \;=\; \bigcup \{\, \chi_{e*}(Lk(w, \square_e)) \mid e \text{ is a cell of } X \text{ and } \chi_e(w) = v \,\}$$

The following definitions are by analogy with the notions of ascending and descending manifolds in Morse theory.

Definition 4.4. The *ascending link* (the $\uparrow$-link) is

$$Lk_{\uparrow}(v,X) \;=\; \bigcup \{\, \chi_{e*}(Lk(w,\Box_e)) \mid \chi_e(w)=v \text{ and } f\chi_e \text{ has a minimum at } w \,\} \;\subset\; Lk(v,X)$$

and the *descending link* (the $\downarrow$-link) is

$$Lk_{\downarrow}(v,X) \;=\; \bigcup \{\, \chi_{e*}(Lk(w,\Box_e)) \mid \chi_e(w)=v \text{ and } f\chi_e \text{ has a maximum at } w \,\} \;\subset\; Lk(v,X).$$

We will use the local link structure to deduce topological information about point preimages or *level sets* of Morse functions. First we fix notaion. Let $f : X \to \mathbb{R}$ be a Morse function as above, and $J \subset \mathbb{R}$ a closed interval. Denote by X_J the preimage $f^{-1}(J)$. When J consists of a single point $t \in \mathbb{R}$ we denote the preimage by X_t and refer to it as a *level set* of f.

The following lemma and corollary are analogues of the basic fact that a Morse function on a manifold gives a handlebody decomposition of the manifold. These are stated and proven in [BB] as Lemma 2.5 and Corollary 2.6 respectively.

Lemma 4.5. *Let $f : X \to \mathbb{R}$ be a Morse function on a PE cubical complex. Suppose $J \subset J'$ are closed and connected intervals in $\mathbb{R}$, $\inf J = \inf J'$, and $J' \setminus J$ contains only one point r of f(0-cells). Then $X_{J'}$ is homotopy equivalent to X_J with the copies of $Lk_{\downarrow}(v,X)$ (v a vertex with $f(v) = r$) coned off.* $\Box$

A similar statement holds when $\inf J = \inf J'$ is replaced by $\sup J = \sup J'$ and $Lk_{\downarrow}(v,X)$ by $Lk_{\uparrow}(v,X)$.

Corollary 4.6. *Let $f : X \to \mathbb{R}$ be a Morse function on a PE cubical complex. Suppose that $J \subset J'$ are intervals in $\mathbb{R}$.*

(1) *If each $\uparrow$-link and each $\downarrow$-link is homologically n-connected, then inclusion $X_J \hookrightarrow X_{J'}$ induces isomorphism in $\tilde{H}_i$ for $i \le n$ and epimorphism in $\tilde{H}_{n+1}$.*
(2) *If each $\uparrow$-link and each $\downarrow$-link is simply connected, then inclusion $X_J \hookrightarrow X_{J'}$ induces isomorphism in π_1.* $\Box$

The theorem below demonstrates the power of the Morse theory point of view. Simple topological conditions on the links give rise to finiteness properties for the kernel subgroup. Most of this result appears as Theorem 4.1 of [BB].

Theorem 4.7. *Let $f : X \to \mathbb{R}$ be a ϕ-equivariant Morse function and let $H = Ker(\phi)$ be as above.*

(1) *Suppose that the reduced homology of each $\uparrow$-link and each $\downarrow$-link is zero in all dimensions 0 through $n+1$, except for dimension n. Then H is of type FP_n but is not of type FP_{n+1}.*
(2) *If all $\uparrow$-links and $\downarrow$-links are simply connected, then H is finitely presented.*

Proof. The main point of this argument is that we have a contractible space $X = f^{-1}((-\infty,\infty))$ expressed as the direct limit of spaces $X_{[t-N,t+N]}$ for $N \in \mathbb{N}$ and a given $t \in \mathbb{R}$.

Now, Corollary 4.6 (1) implies that $\tilde{H}_i(X_{[t-N,t+N]},\mathbb{Z})$ is isomorphic to $\tilde{H}_i(X_t,\mathbb{Z})$ for $0 \leq i \leq n-1$, and all $N \in \mathbb{N}$. Since homology commutes with direct limits, this implies that $\tilde{H}_i(X_t,\mathbb{Z}) = 0$ for $0 \leq i \leq n-1$. We now show that $\tilde{H}_n(X_t,\mathbb{Z})$ is not finitely generated as a $\mathbb{Z}H$-module. Part (1) then follows immediately from Proposition 4.2 (1).

Suppose to the contrary, that the $\mathbb{Z}H$-module $\tilde{H}_n(X_t,\mathbb{Z})$ has a finite basis, with cycle representatives

$$z_1, \ldots, z_k\,.$$

Then there exists an integer $N \in \mathbb{N}$ such that each of the z_i bound (are null homologous) in $X_{[t-N,t+N]}$. Since H acts by "horizontal" translations on X, this implies that the inclusion induced map

$$\tilde{H}_n(X_t,\mathbb{Z}) \;\to\; \tilde{H}_n(X_{[t-N,t+N]},\mathbb{Z})$$

is trivial. But by Corollary 4.6 (1) it is also an epimorphism. Thus $\tilde{H}_n(X_{[t-N,t+N]},\mathbb{Z}) = 0$. Let $J \subset \mathbb{R}$ be a closed interval such that $[t-N,t+N] \subset J$ and $\mathbb{Z} \cap (J \setminus [t-N,t+N]) \neq \emptyset$. Lemma 4.5 implies that X_J can be obtained (upto homotopy) from $X_{[t-N,t+N]}$ by coning off successive copies of the ascending and descending links of f. Let L denote such a link. The Mayer-Vietoris sequence together with the hypothesis $\tilde{H}_n(L,\mathbb{Z}) \neq 0$ implies that $\tilde{H}_{n+1}(X_J,\mathbb{Z}) \neq 0$ for all such intervals J. Now the hypothesis $\tilde{H}_{n+1}(L,\mathbb{Z}) = 0$ together with the Mayer-Vietoris sequence shows that the inclusion induced maps

$$\tilde{H}_{n+1}(X_J,\mathbb{Z}) \;\to\; \tilde{H}_{n+1}(X_{J'},\mathbb{Z})$$

are injective for all intervals $J \subset J' \subset \mathbb{R}$. This implies that $\tilde{H}_{n+1}(X,\mathbb{Z}) \neq 0$ a contradiction.

Now for part (2). Since the links are 1-connected, they are connected. The argument in (1) above implies that the level sets X_t are also connected. Corollary 4.6 (2) implies that the inclusion $X_t \hookrightarrow X$ induces an isomorphism in π_1. Hence X_t is 1-connected, and the finite presentability of H follows from Proposition 4.2 (2). $\square$

§5. Branched Coverings of Piecewise Euclidean Cubical Complexes

We will construct the example in §6 by taking a "branched covering" of a particular PE cubical complex. In this section, we define and study the operation of taking a branched covering of a finite PE cubical complex.

We develop the notion of branched covering by analogy with the branched covering of a 3-manifold over a link. The first problem one notices is that a subcomplex of a PE cubical complex need not have a nice neighborhood like a link in a 3-manifold. We consider the following subcomplexes.

Definition 5.1. Let K be a PE cubical complex, and $L \subset K$ a subcomplex. We say that L is a *branching locus* if it satisfies the following two conditions.

• For each m-cell $\chi_e : \square_e^m \to e$ of K with $e \cap L \neq \emptyset$ we have that $\chi_e^{-1}(L)$ is a disjoint union of faces of $\square_e^m$ (possibly all of $\square_e^m$).

• $Lk(e, K) \setminus Lk(e, L)$ is nonempty and connected for each cell $e \subset L$.

The first condition is a local convexity condition for L in K. It is used to describe the neighborhoods of L in K, and to prove that nonpositive curvature is preserved on passing to branched coverings in Lemma 5.5 below.

The second condition is a reformulation of the "codimension 2" condition in branched coverings of manifolds over submanifolds. It ensures that the trivial branched covering of a PE cubical complex K over a branching locus L is just K itself.

Definition 5.2. Let K be a PE cubical complex and $L \subset K$ be a branching locus as above. By a *finite branched covering* of K over L we mean the result of the following process.

• Take a finite covering of $K \setminus L$.

• Lift the local piecewise euclidean metric on $K \setminus L$ to this cover, and then consider the induced path metric.

• Finally, take the metric completion of this cover.

Let X be a metric space and $Y \subset X$. By an ϵ-neighborhood of Y in X—denoted by $N_\epsilon(Y, X)$—we mean the set of all points of X which are within ϵ of Y.

For $L \subset K$ as above and C a component of L, we see that the neighborhoods $N_\epsilon(C, K)$ are all disjoint for $0 < \epsilon < 1/2$. We can describe these neighborhoods as follows.

Suppose that $C \subset L$ is a component of L, and $\chi_e : \square_e^m \to e$ is an m-cell of K such that $\emptyset \neq \chi_e^{-1}(C)$ is a proper face of $\square_e^m$. Then, for $0 < \epsilon < 1/2$ there is a natural continuous map

$$f_e : \partial N_\epsilon(\chi_e^{-1}(C), \square_e^m) \ \to \ \chi_e^{-1}(C)$$

defined by nearest point retraction. Note that $N_\epsilon(\chi_e^{-1}(C), \square_e^m)$ is homeomorphic to the mapping cylinder of f_e. The composites $\chi_e f_e$ paste together to define a continuous map

$$f_C : \partial N_\epsilon(C, K) \ \to \ C$$

and the neighborhood $N_\epsilon(C, K)$ is homeomorphic to the mapping cylinder $M(f_C)$.

We say that the branched covering is *trivial* if the finite cover in step 2 of the definition above is a trivial cover. The following Lemma shows that the operation of finite branched coverings does not take one out of the class of finite PE cubical complexes.

Lemma 5.3. *Let K be a finite PE cubical complex, and let $\hat{K}$ be a finite branched covering. Then $\hat{K}$ is also a finite PE cubical complex, and there is a natural continuous surjective map $b : \hat{K} \to K$. In the case of a trivial branched covering, one has $\hat{K} = K$.*

Proof. Note that $K \setminus L$ is the union of a finite collection of "cubes" (that is regular cubes, possibly with some lower dimensional faces removed) glued together by isometries of their faces. Each of these "cubes" is 1-connected and so lifts to the finite cover $\overline{K \setminus L}$ of $K \setminus L$. Thus, this finite cover is a finite union of such "cubes" glued together by isometries.

Now, the path metric on the finite covering is locally Euclidean. Thus, on forming the completion one just adds faces back to the "cubes" in $\overline{K \setminus L}$, and so $\hat{K}$ inherits the structure of a finite PE cubical complex.

We shall show that the covering map $\overline{K \setminus L} \to K \setminus L$ extends to a continuous surjective map $b : \hat{K} \to K$.

Since the covering map $\overline{K \setminus L} \to K \setminus L$ is locally distance nonincreasing, the induced map $b : \hat{K} \to K$ is well-defined and continuous.

Let x be a point in a cell $e \subset L$. Since $Lk(e, K) \setminus Lk(e, L)$ is nonempty, there is a cell $e_1 \subset K$ such that $e \subset e_1$ and $\chi_{e_1}^{-1}(L)$ is contained in the boundary of $\square_{e_1}$. Any path in the interior of e_1 which ends at x lifts to a path in the finite cover. Thus a Cauchy sequence in this path which converges to x lifts to a Cauchy sequence defining a point $\hat{x} \in \hat{K}$ such that $b(\hat{x}) = x$. Therefore b is surjective.

Finally, let s_1 and s_2 denote two Cauchy sequences of points in $K \setminus L$, which are Cauchy equivalent in K and which define a point $x \in L$. Since $\emptyset \neq (Lk(e, K) \setminus Lk(e, L))$ is connected for each cell $e \subset L$, then s_1 and s_2 will still be Cauchy equivalent in the induced path metric of $K \setminus L$. Thus the completion of the trivial cover $K \setminus L$ yields the complex K as required. $\square$

If C is a component of the branching locus L and $0 < \epsilon < 1/2$. Then the deleted ϵ-neighborhood $N_\epsilon(C, K) \setminus C$ of C lifts to the finite cover to give deleted ϵ-neighborhoods of components of the preimage of C in $\hat{K}$.

The next lemma derives a consequence of the local convexity condition in the definition of branching locus. It will be used in the proof of Lemma 5.5 below.

Lemma 5.4. *Let K be a finite PE cubical complex of nonpositive curvature, $L \subset K$ a branching locus, and $v \in L$ a vertex. Then $Lk(v, L)$ is a full subcomplex of $Lk(v, K)$.*

Proof. Since K is of nonpositive curvature the links $Lk(v, K)$ are simplicial complexes (flag complexes in fact). It remains to see that if a collection of vertices in $Lk(v, L)$ spans a simplex σ in $Lk(v, K)$ then $\sigma \subset Lk(v, L)$.

Let $\{v_1, \dots, v_m\}$ be a collection of vertices in $Lk(v, L)$ which spans a simplex σ in $Lk(v, K)$. This simplex determines a cell e_σ at the vertex v in K. Since $v \in L$ we have $e_\sigma \cap L \neq \emptyset$, and since each $v_i \in Lk(v, L)$ we see that all the edges of e_σ at the vertex v are contained in L. The local convexity condition implies that $\chi_{e_\sigma}^{-1}(L) = \square_{e_\sigma}$ and so $e_\sigma \subset L$. Thus $\sigma \subset Lk(v, L)$ as required. $\square$

Finally, we show that a finite branched covering of a PE cubical complex of nonpositive curvature with branching locus a finite graph is a PE cubical complex of nonpositive curvature. This will be important in Section 6 as we will need to know that the universal cover of such a finite branched covering is a CAT(0) metric space.

Lemma 5.5. *Let $b : \hat{K} \to K$ be a finite branched covering of a PE cubical complex K of nonpositive curvature, and suppose that the branching locus $L \subset K$ is a finite graph. Then $\hat{K}$ is also a nonpositively curved PE cubical complex.*

Proof. We know from Lemma 5.3 that $\hat{K}$ is a finite PE cubical complex. It remains to verify that the links are all flag complexes.

Let $v \in \hat{K}$ be a vertex. First we show that $Lk(v, \hat{K})$ is a simplicial complex. If not, then there exist simplices $\sigma, \tau \subset Lk(v, \hat{K})$ such that $\sigma \neq \tau$ yet the boundary of σ equals the boundary of τ. The map b induces a map of links $b_* : Lk(v, \hat{K}) \to Lk(b(v), K)$. Since $Lk(b(v), K)$ is a simplicial complex, the collection of vertices of σ is mapped injectively by b_* to a collection of veritces in $Lk(b(v), K)$. Furthermore, we have $b_*(\sigma) = b_*(\tau)$ in $Lk(b(v), K)$.

Now each vertex $w \in \sigma$ determines an edge $e = e_w$ at v in $\hat{K}$. If $b(e) \not\subset L$ then the induced map of links $b_* : Lk(e, \hat{K}) \to Lk(b(e), K)$ is an isomorphism. Since $Lk(e, \hat{K})$ may be identified with $Lk(w, Lk(v, \hat{K}))$ this implies that the map of links $b_* : Lk(v, \hat{K}) \to Lk(b(v), K)$ is locally injective at $w \in Lk(v, \hat{K})$. But this contradicts the fact that $\sigma \neq \tau$ yet $b_*(\sigma) = b_*(\tau)$.

Thus, the vertex set of σ maps injectively to the vertex set of $b_*(\sigma) \subset Lk(b(v), K)$, and each vertex of $b_*(\sigma)$ is contained in $Lk(b(v), L)$. By Lemma 5.4 $Lk(b(v), L)$ is a full subcomplex of $Lk(b(v), K)$ and so $b_*(\sigma) \subset Lk(b(v), L)$.

So b_* maps both σ and τ injectively to the simplex $b_*(\sigma) \subset Lk(b(v), L)$. But L is a finite graph, and so $b_*(\sigma)$ is either empty or a single point. Thus $\sigma = \tau$ and this contradiction shows that $Lk(v, \hat{K})$ is a simplicial complex.

Finally, we show that $Lk(v, \hat{K})$ is a flag complex. Let $\{v_1, \dots, v_n\}$ be a collection of pairwise adjacent vertices in $Lk(v, \hat{K})$. Since $Lk(b(v), K)$ is a simplicial complex, this collection maps bijectively to the collection $\{b_*(v_1), \dots, b_*(v_n)\}$ of pairwise adjacent vertices in $Lk(b(v), K)$. Now, since $Lk(b(v), K)$ is a flag complex, the $\{b_*(v_1), \dots, b_*(v_n)\}$ spans an $(n-1)$-simplex. This determines an n-cell based at $b(v)$ in K. This "lifts" to a cell of $\hat{K}$ which is based at v. Thus the original collection spans an $(n-1)$-simplex in $Lk(v, \hat{K})$ as required. □

It appears that more subtle conditions are necessary in order to preserve nonpositive curvature when the branching locus contains cells of dimension 2 or higher. We are currently investigating this question.

§6. The Example

In this Section we construct a PE cubical complex Y and a continuous map $f : Y \to S^1$ satisfying the conditions in the following theorem.

Theorem 6.1. *There exists a piecewise euclidean cubical complex Y and a continuous map*
$f : Y \to S^1$ with the following properties.

1. *The image of $\phi = f_* : \pi_1(Y) \to \pi_1(S^1)$ is of finite index in $\pi_1(S^1)$.*
2. *The universal cover $X = \tilde{Y}$ is a CAT(0) metric space.*
3. *X has no embedded flat planes.*

(4) *The map f lifts to a ϕ-equivariant Morse function $\tilde{f} : X \to \mathbb{R}$ whose ascending and descending links are all homeomorphic to S^2.*

First we show that Theorem 1.1 follows from Theorem 6.1 above.

Proof of Theorem 1.1. Part (1) implies that there is a short exact sequence of groups

$$1 \to H \to G \to \mathbb{Z} \to 1$$

where $G = \pi_1(Y)$, and $\mathbb{Z} = \phi(G)$ is a finite index subgroup of $\pi_1(S^1)$.

Now G acts freely, properly, cocompactly by isometries on the CAT(0) metric space X by property (2). Property (3) and Theorem 2.2 now imply that X is a δ-hyperbolic metric space, and we conclude from Theorem 2.1 that G is a torsion free hyperbolic group.

Theorem 4.7 and part (4) above imply that H is finitely presented but not of type FP_3.

Finally, if H were hyperbolic, Theorem 2.3 would imply that its Rips complex is a finite $\mathrm{K}(H, 1)$. In particular, H would be of type FP_3 (in fact H would be of finite type), which contradicts the conclusion above. □

Proof of Theorem 6.1. The construction of the PE cubical complex Y and continuous map $f : Y \to S^1$ proceeds as follows.

We begin with a PE cubical complex Θ^3 and a continuous map $d : \Theta^3 \to S^1$ such that the image of $d_* : \pi_1(\Theta^3) \to \pi_1(S^1)$ is a finite index subgroup of $\pi_1(S^1)$, and such that d lifts to a Morse function on the universal cover whose ascending and descending links are all isomorphic to the 2-sphere with simplicial structure given as the join of three 0-spheres (i.e. a regular octahedron). The problem with Θ^3 is that $\pi_1(\Theta^3)$ will contain many $\mathbb{Z} \oplus \mathbb{Z}$ subgroups, and so is not hyperbolic.

Next we take a branched covering Y of Θ^3 so that the universal cover of Y is a δ-hyperbolic metric space. The map $f : Y \to S^1$ is just defined to be the composition of this branched covering map and the map $d : \Theta^3 \to S^1$. If one is careful in the choice of branched covering, the map f will lift to the universal covers to give a Morse function whose ascending and descending links are all homeomorphic to S^2 (though not necessarily isomorphic to an octahedron).

Here are the details of these steps.

Step I: The complex Θ^3 and the map $d : \Theta^3 \to S^1$

We denote by Θ the graph consisting of two vertices and four edges, obtained from four copies of the unit interval by identifying all the 0-endpoints and identifying all the 1-endpoints. Define a continuous map

$$c : \Theta \to S^1$$

so that its restriction to each of two of the edges of Θ is given by $x \mapsto [x]$, and its restriction to each of the other two edges is given by $x \mapsto [1 - x]$. Here $[x]$ denotes the equivalence class of x in $\mathbb{R}/\mathbb{Z}$. Note that c maps both vertices of Θ to the basepoint $[0] \in S^1$. We now take the map c on each factor to obtain a continuous map map $c^3 : \Theta^3 \to T^3$.

The linear map $l : \mathbb{R}^3 \to \mathbb{R} : (x, y, z) \mapsto x + y + z$ induces a continuous map of quotient spaces $T^3 \to S^1$ which we also denote by l. Composing l with c^3 above gives our map

$$d \ = \ c^3 \circ l : \Theta^3 \to S^1$$

Note that Θ^3 consists of sixty-four 3-cubes, all sharing the same eight vertices. The cubes are glued together along faces, edges, and vertices, and so Θ^3 has a natural PE cubical complex strucutre. The link of each vertex is an all-right piecewise spherical complex, which is isomorphic to the 3-fold join of a set of four points. Thus, all the links are flag complexes, and so Θ^3 is a PE cubical complex of nonpositive curvature.

The map c lifts to a map of universal covers $\tilde{c} : \widetilde{\Theta} \to \mathbb{R}$ which is easily seen to be a ϕ-equivariant Morse function whose ascending and descending links are all isomorphic to S^0. Thus $d = c^3 \circ l$ lifts to a map

$$\tilde{d} = l \circ \tilde{c}^3 : \widetilde{\Theta}^3 \to \mathbb{R}^3 \to \mathbb{R}$$

Now the restriction of $\tilde{d}$ to a 3-cube of $\widetilde{\Theta}^3$ is just given by the restriction of the linear map $l : \mathbb{R}^3 \to \mathbb{R}$ to a 3-cube, possibly postcomposed with an integer translation $\mathbb{R} \to \mathbb{R}$.

Thus we see that $\tilde{d}$ is a ϕ-equivariant Morse function on $\widetilde{\Theta}^3$. Its ascending and descending links are all isomorphic to the 2-sphere with simplicial structure given as the 3-fold join of S^0.

Now $\widetilde{\Theta}^3$ is a CAT(0) metric space in the induced path metric, and so Theorem 4.7 implies that $\mathrm{Ker}(d_* : \pi_1(\Theta^3) \to \mathbb{Z})$ is finitely presented but not of type FP_3.

However, $\pi_1(\Theta^3)$ contains $\mathbb{Z}^3$ subgroups and so is not a hyperbolic group. We now modify this example to obtain a cubical complex Y whose fundamental group is hyperbolic.

Step II: The branched covering $Y \to \Theta^3$

The idea here is to take a suitable branched covering $Y \to \Theta^3$ so that $\pi_1(Y)$ is a hyperbolic group. However, one must be careful, because branched coverings will affect the local link structure, which in turn may affect the Morse theory argument and the finiteness properties of the kernel subgroup. For instance, after branched coverings the ascending links may no longer be 1-connected. This new link structure would give rise to kernels which might not be finitely presented.

We will take a branched covering of Θ^3 with the follong branching locus L.

$$L \ = \ (0 \times 1 \times \Theta) \ \cup \ (\Theta \times 0 \times 1) \ \cup \ (1 \times \Theta \times 0) \ \ \subset \ \ \Theta^3$$

Properties (1) and (2) of Theorem 6.1 hold for all finite branched covers of Θ^3 over L. After verifying this, we shall construct an explicit finite branched cover of Θ^3, and then show that properties (3) and (4) hold for this covering.

Proof of Theorem 6.1 part (1). In this example, the components of the branching locus L are copies of the graph Θ and so contain loops which map onto the circle S^1 with degree two. A parallel copy $(\{1/3\} \times 1 \times \Theta)$ of the component $(0 \times 1 \times \Theta)$ of L also contains loops which map by degree two onto S^1. This parallel copy is contained in $\Theta^3 \setminus L$ and its preimage in the branched covering $b : Y \to \Theta^3$ contains a loop which maps onto S^1 with nontrivial finite degree. This degree gives an upper bound for the index of $\phi(\pi_1(Y))$ in $\pi_1(S^1)$. $\square$

Proof of Theorem 6.1 part (2). The complex Y is obtained as a branched covering of the nonpositively curved PE cubical complex Θ^3. The branching locus L satisfies the conditions in Lemma 5.3 and Lemma 5.5. Thus Y is a PE cubical complex of nonpositive curvature, and so its universal cover X will be a CAT(0) metric space. $\square$

Now we choose an explicit finite branched covering

$$b : Y \ \to \ \Theta^3$$

keeping properties (3) and (4) in mind. We begin by considering Θ as a labelled, directed graph as indicated in Figure 1. The edges a and b map to the circle by the map $x \mapsto [x]$ and they are directed from 0 to 1. The edges c and d map via $x \mapsto [1-x]$, and they are directed from 1 to 0. As above $[x]$ denotes the equivalence class of the real number x in $S^1 = \mathbb{R}/\mathbb{Z}$.

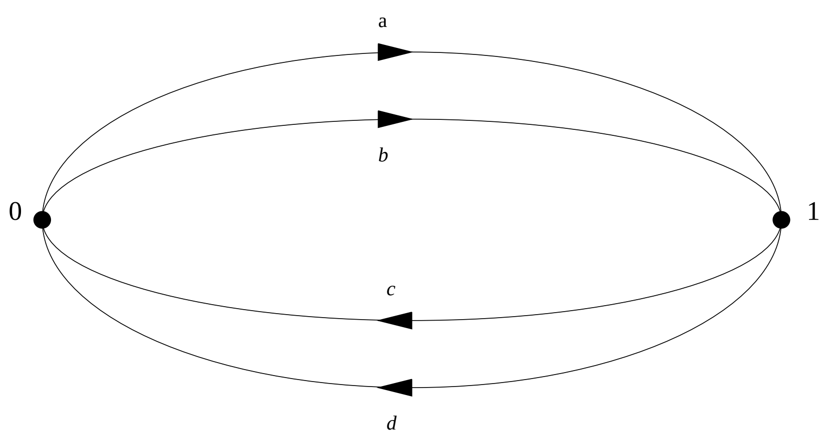

FIGURE 1. The graph Θ

Definition 6.1. Denote by Δ the graph

$$\Theta \times \{0\} \ \cup \ \{1\} \times \Theta \quad \subset \quad \Theta^2 \setminus \{(0,1)\}$$

Note that Δ is a graph with three vertices and eight edges. The edges inherit their orientation from Θ. We choose the following labelling (see Figure 2)

$$x_0 \quad \text{denotes} \quad x \times \{0\} \quad \text{for} \quad x \in \{a, b, c, d\}$$

and

$$x_1 \quad \text{denotes} \quad \{1\} \times x \quad \text{for} \quad x \in \{a, b, c, d\}$$

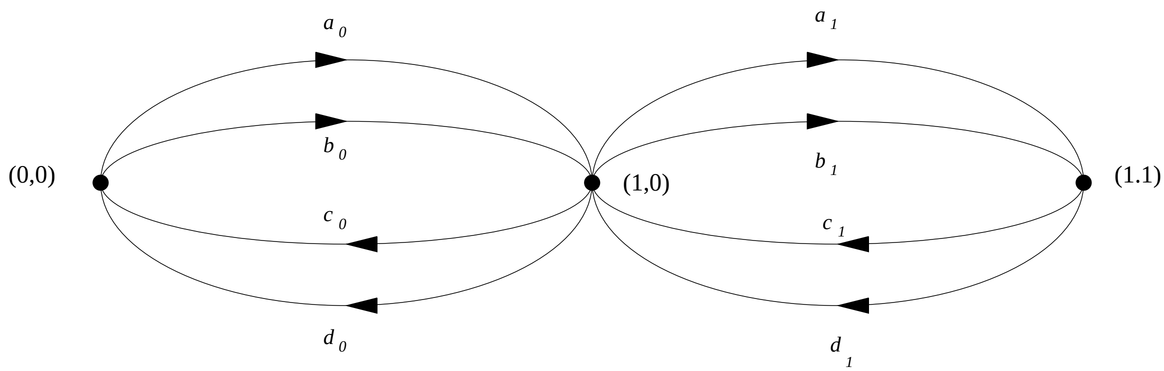

FIGURE 2. The graph Δ

Remark. The fundamental group $\pi_1(\Delta, (1,0))$ is free of rank six. We can take as basis the loops:

$$\overline{a_0}b_0\,, \quad \overline{a_0 c_0}\,, \quad \overline{a_0}\overline{d_0}\,, \quad a_1\overline{b_1}\,, \quad a_1c_1\,, \quad a_1d_1$$

Lemma 6.2. *The inclusion $\Delta \subset \Theta^2 \setminus \{(0,1)\}$ is a homotopy equivalence.*

Proof. We exhibit a strong deformation retraction of $\Theta^2 \setminus \{(0,1)\}$ onto Δ.

Note that Θ^2 can be obtained from four 2-tori as follows. Each torus is subdivided into four squares by two parallel meridians and two parallel longitudes. Two tori are glued together along a pair of parallel meridians or a pair of parallel longitudes. The glueing pattern can be read from Θ^2 once we view each Θ as a union of two circles identified along a 0-sphere.

Each torus may be described as the quotient of a square in the plane, bounded by the lines $y = \pm 1$ and $x = \pm 1$, by identifying points which differ by either a horizontal or a vertical translation through two units. Suppose that the origin maps to the vertex $(0,1) \in \Theta^2$. Then radial retraction (along geodesics) from each such square minus the origin to its boundary induces the usual strong deformation retraction from a punctured torus to a figure eight graph, and these paste together to give the deformation retraction $\Theta^2 \setminus \{(0,1)\} \to \Delta$. $\square$

Recall that the branching locus is

$$L \;=\; (0 \times 1 \times \Theta) \;\cup\; (\Theta \times 0 \times 1) \;\cup\; (1 \times \Theta \times 0)$$

Definition 6.3. Define three *projections*

$$\mathrm{pr}_i : \Theta^3 \setminus L \;\to\; \Theta^2 \setminus \{(0,1)\} \qquad (i = 1,2,3)$$

by

$$\mathrm{pr}_1 : (x,y,z) \mapsto (y,z)$$
$$\mathrm{pr}_2 : (x,y,z) \mapsto (z,x)$$
$$\mathrm{pr}_3 : (x,y,z) \mapsto (x,y)$$

Remark. Note that the maps pr_i are all continuous onto. Note also that we have well defined homomorphisms

$$\mathrm{pr}_{i*} : \pi_1(\Theta^3 \setminus L\,,\,(0,0,0)) \;\to\; \pi_1(\Theta^2 \setminus \{(0,1)\}\,,\,(0,0))$$

We recall that a finite covering may be prescribed by a representation of the fundamental group of the base into a finite symmetric group. We shall choose a representaion of $\pi_1(\Delta)$ into the symmetric group S_5, and use the homotopy equivalence of Δ with $\Theta^2 \setminus \{(1,0)\}$ given in Lemma 6.2 together with the maps $\mathrm{pr}_i : \Theta^3 \setminus L \to \Theta^2 \setminus \{(1,0)\}$ to describe a representation of $\pi_1(\Theta^3 \setminus L, (0,0,0))$ as follows.

Begin by defining a representation

$$\rho : \pi_1(\Delta, (1,0)) \;\to\; \mathrm{S}_5$$

which maps the free basis as follows

$$\overline{a_0}b_o \mapsto \alpha\,,\;\; \overline{a_0 c_0} \mapsto \alpha^2\,,\;\; \overline{a_0}\overline{d_0} \mapsto \alpha^3$$

$$a_1\overline{b_1} \mapsto \beta\,,\;\; a_1 c_1 \mapsto \beta^2\,,\;\; a_1 d_1 \mapsto \beta^3$$

where $\alpha = (2354) \in \mathrm{S}_5$ and $\beta = (12345) \in \mathrm{S}_5$.

The permutations α and β are chosen to satisfy the relation

$$\alpha\beta\alpha^{-1} \;=\; \beta^2\,.$$

Therefore $\alpha^i\beta^j\alpha^{-i} = \beta^{2^i j}$, and so the following commutation relations hold

$$[\alpha^i, \beta^j] \;=\; \beta^{(2^i-1)j} \qquad \text{for integers } i,j.$$

Note that for $i \in \{1,2,3\}$ and $j \in \{1,2,3,4\}$ the commutator $[\alpha^i, \beta^j]$ is a 5-cycle (in fact a nontrivial power of β). The corresponding 5-fold covering has the following important property.

Property 1. There are $\binom{4}{2}$ choices of a pair of edges between $(0,0)$ and $(1,0)$ in Δ. Each such pair determines a loop (or its inverse) based at $(1,0)$ and hence an element of $\pi_1(\Delta,(1,0))$. Likewise, there are six elements of $\pi_1(\Delta,(1,0))$ determined by pairs of edges between $(1,0)$ and $(1,1)$. The first six elements are simple products of the first three basis elements of $\pi_1(\Delta,(1,0))$, and so are mapped to α^i for $i \in \{1,2,3\}$ by ρ. The second six are products of the last three basis elements and map to β^j for $j \in \{1,2,3,4\}$ under ρ.

One can form 36 commutators by pairing an element from the first six above with an element from the second six. Each such commutator maps to a nontrivial power of β, and so the preimage of the corresponding loop in $\overline{\Delta}$ consists of a single component which maps by degree 5 onto the commutator loop. This property will be crucial in establishing parts (3) and (4) of Theorem 6.1.

For each $i \in \{1,2,3\}$ we define the homomorphism ρ_i to be the composite

$$\pi_1(\Theta^3 \setminus L,(0,0,0)) \overset{\mathrm{pr}_i}{\to} \pi_1(\Theta^2 \setminus \{(0,1)\}\,,(0,0)) \overset{\cong}{\to} \pi_1(\Theta^2 \setminus \{(0,1)\}\,,(1,0)) \overset{\rho}{\to} \mathrm{S}_5$$

The middle arrow is just a change of basepoint isomorphism. The arrow labelled ρ involves a slight abuse of notation. It is actually the map ρ precomposed with the inverse of the inclusion induced isomorphism $\pi_1(\Delta,(1,0)) \to \pi_1(\Theta^2 \setminus \{(0,1)\},(1,0))$.

These homomorphisms ρ_i define three different actions of $\pi_1(\Theta^3 \setminus L\,,(0,0,0))$ on the set $S = \{1,2,3,4,5\}$ and can be combined to define an action on the set $S \times S \times S$. We use this homomorphism

$$\tau = (\rho_1,\rho_2,\rho_3) \,:\, \pi_1(\Theta^3 \setminus L\,,(0,0,0)) \;\to\; \mathrm{S}_{125}$$

to define a 125-fold cover of $\Theta^3 \setminus L$. We lift the local metric from $\Theta^3 \setminus L$ and complete to obtain the branched covering

$$b \,:\, Y \;\to\; \Theta^3$$

Before proving parts (3) and (4) of Theorem 6.1 we give a description of the links of vertices in Y. A vertex $v \in Y$ is said to be of *type I* if $b(v) \in \{(0,0,0),(1,1,1)\}$ and is said to be of *type II* otherwise.

Lemma 6.4. *Let $b : Y \to \Theta^3$ be the branched covering defined above, and let $v \in Y$ be a vertex. Let F be a 0-complex consisting of just four vertices.*

(1) *If v is of type I, then $Lk(v,Y)$ is isomorphic to the 3-fold join $F * F * F$.*
(2) *If v is of type II, then $Lk(v,Y)$ is isomorphic to the join $\Gamma * F$, where $\Gamma \to F * F$ is a finite (5-fold) covering in which each of the thirty six loops of the form $S^0 * S^0 \subset F * F$ has connected preimage in Γ.*

Proof. The vertices $(0,0,0)$ and $(1,1,1)$ do not lie on the branching locus L and so neighborhoods of lifts of these vertices in Y will be isomorphic to the neighborhoods of $(0,0,0)$ or $(1,1,1)$ in Θ^3. Thus, if $v \in Y$ is a vertex of type I, then $Lk(v,Y)$ is isomorphic (as a simplicial complex) to the 3-fold join $F * F * F$.

A vertex $v \in Y$ of type II is the lift of a vertex in Θ^3 which lies on a component of the branching locus L. This component has a deleted neighborhood of the form

$$\Theta \times (F * F) \times (0, \epsilon)\,.$$

This lifts to a subset of the finite cover $\overline{\Theta^3 \setminus L} \to \Theta^3 \setminus L$ of the form

$$\overline{\Theta} \times \Gamma \times (0, \epsilon)\,.$$

On forming the completion one adds a copy of the graph $\overline{\Theta}$ which contains the vertex v. Thus, the link of v is of the form

$$Lk(v, Y) \;\cong\; \Gamma * F$$

and so we need only check that $\Gamma \to F * F$ has the appropriate covering properties.

To determine the covering $\Gamma \to F * F$ we consider a specific case. Suppose the vertex v maps to the vertex $(0,0,1) \in \Theta^3$. Note that $(0,0,1)$ lies on $\Theta \times 0 \times 1 \subset L$. We view the $F * F$ above as the boundary of an ϵ-neighborhood of $(0,0,1)$ in the 2-complex $0 \times \Theta^2$. As such, it is a subset of $\Theta^3 \setminus L$ and so determines a subgroup of $\pi_1(\Theta^3 \setminus L, (0,0,0))$ up to conjugacy. The conjugacy class depends on choice of path from $(0,0,0)$ to the graph $F * F$. The images $\mathrm{pr}_i(F * F)$ are contractible for $i = 2, 3$, while pr_1 maps $F * F$ isomorphically onto the boundary of an ϵ-neighborhood of $(0,1)$ in Θ^2 (this is a subset of $\Theta^2 \setminus \{(0,1)\}$).

Thus the representation ρ induces an action of the subgroup of $\pi_1(\Theta^3 \setminus L, (0,0,0))$ determined by $F * F$ on the set $S \times S \times S$ which is trivial on the 2nd ans 3rd factors. Moreover each of the 36 loops of the form $S^0 * S^0 \subset F * F$ map to the commutators defined in Property 1 above, and so act on the first factor by 5-cycles. Thus the preimage of $F * F$ in the 125-fold cover $\overline{\Theta^3 \setminus L} \to \Theta^3 \setminus L$ determined by ρ consists of 25 components of the form $\Gamma \to F * F$. Each component is a 5-fold covering in which all 36 loops $S^0 * S^0 \subset F * F$ have connected preimages. The other cases of type II vertices are handled similarly. $\square$

Proof of Theorem 6.1 part (3). We argue by contradiction. Suppose that $i : \mathbb{E} \to X$ is an isometric embedding of a flat plane. First, we shall show that $i(\mathbb{E})$ intersects a lift of the branching locus transversely at certain points. Secondly, a local argument implies that the angle sum about such a point of $\mathbb{E}$ exceeds 2π, thus contradicting the fact that $\mathbb{E}$ is a flat plane.

We say that a point $x \in \mathbb{E}$ is a *transverse* intersection point with a lift of L if there exists an $\epsilon > 0$ such that the ϵ-ball about x in $\mathbb{E}$ intersects the i-preimage of the lift of L only in the point x. We say that the flat $i : \mathbb{E} \to X$ intersects that lift of L transversally.

Step 1: (Existence of transverse intersection point). Denote by $\tilde{L}$ the preimage of L in X. Each 3-cell of X intersects $\tilde{L}$ in three disjoint, nonparallel edges as shown in Figure 3.

Now $i(\mathbb{E})$ is an isometric embedding, and so is not contained in the 1-skeleton of X. Hence, there exists a 3-cell $e \subset X$ such that $i(\mathbb{E}) \cap e$ is 2-dimensional. That is, the set $\chi_e^{-1}(i(\mathbb{E}) \cap e)$ is a nontrivial polygon of intersection of a 2-plane with a standard 3-cube in $\mathbb{R}^3$. Here χ_e is an admissible characteristic map for e.

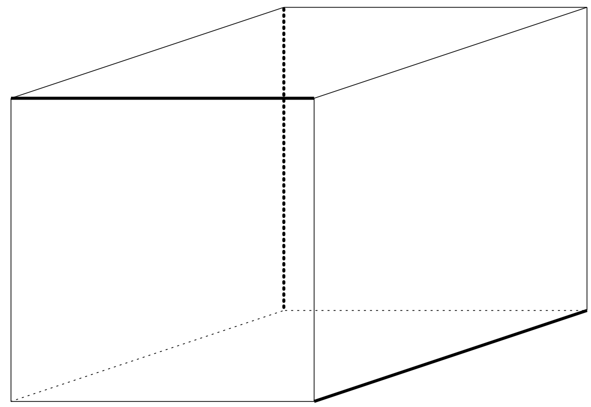

FIGURE 3. Intersection pattern of $\tilde{L}$ on a 3-cube of X

There are two cases to consider. Firstly, $i(\mathbb{E})\cap(\tilde{L}\cap e) \neq \emptyset$. In this case, the configuration in Figure 3 shows us that $\chi_e^{-1}(i(\mathbb{E})$ must intersect at least one edge of $\chi_e^{-1}(\tilde{L})$ in a single point p. Moreover, this edge is not contained in the plane of $\mathbb{R}^3$ defined by $\chi_e^{-1}(\mathbb{E})$. We claim that $x = i^{-1}(\chi_e(p)) \in \mathbb{E}$ is a transverse intersection point.

If p is an interior point of the edge of $\chi_e^{-1}(\tilde{L})$ then it clearly determines a transverse intersection point $x \in \mathbb{E}$. One just works with an ϵ-ball about x, where ϵ is the minimum distance from p to the endpoints of the edge. If p is an endpoint of the edge, then any $\epsilon < 1$ works in the definition of transverse intersection point. We see this as follows.

If x is not a transverse intersection point, then $i(\mathbb{E})$ must intersect the graph $\tilde{L}$ in points arbitrarily close to the vertex $\chi_e(p)$. Now $i : \mathbb{E} \to X$ is an isometric embedding, and so if $i(\mathbb{E})$ contains two points on an edge of X it must contain the whole edge. By our choices, $i(\mathbb{E})$ does not contain the edge of $\tilde{L} \cap e$ at $\chi_e(p)$, and so must contain one of the other three edges of $\tilde{L}$ at the vertex $\chi_e(p)$. This edge lies in another 3-cell $e' \subset X$ which shares a 2-dimensional face with e. But the union of these two cells develops into $\mathbb{R}^3$, and $i(\mathbb{E}) \cap (e \cup e')$ develops into a subset which contains the union of a polygon in the first cube and an edge of the second. Since the edge does not lie in the plane of $\mathbb{R}^3$ defined by the polygon, we contradict the fact that i is an isometric embedding.

Secondly, $i(\mathbb{E})\cap(\tilde{L}\cap e) = \emptyset$. In this case, $i(\mathbb{E})$ must intersect an edge of e which contains a vertex v of type I. Developing $i(\mathbb{E})$ in the neighborhood of v we obtain a 2-dimensional hyperplane in the union of 8 unit cubes about the origin in $\mathbb{R}^3$ as shown in Figure 4. We choose the admissible characteristic maps so that v maps to the origin and the collection of 8 cubes at v develop out to the unit cubes based at $(0, 0, 0)$, whose union is the cube Q of sidelength 2 with vertex set $\{-1, 1\}^3 \subset \mathbb{R}^3$. Up to composition with an isometry of $\mathbb{R}^3$, the intersection pattern with $\tilde{L}$ will develop out as shown in Figure 4. (Note that we have just developed a subset of the neighborhood of v.)

It is easy to see from Euclidean geometry that a plane which intersects the union of the coordinate axes within Q must also transversally intersect one of the 6 lines $z = 0, y = \pm 1$;

$y = 0, x = \pm 1$; $x = 0, z = \pm 1$ within Q. Now we argue as in the previous case to see that such intersection points determine transverse intersection points $x \in \mathbb{E}$.

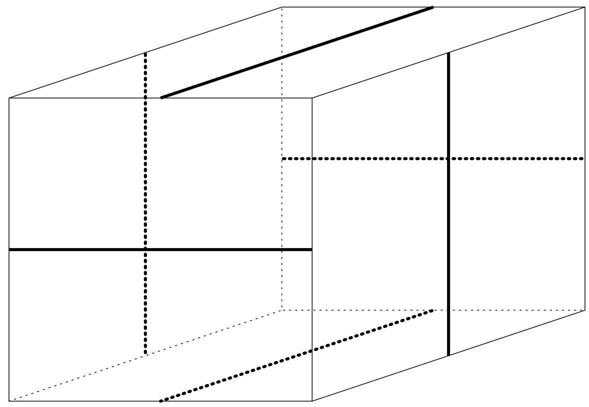

FIGURE 4. Intersection pattern of $\tilde{L}$ on an $\mathbb{R}^3$ neighborhood of a type I vertex

Step 2: (Local angle computation at transverse intersection point). Suppose that $x \in \mathbb{E}$ is a transverse intersection point. Then the i-preimage of a 3-cell of X which contains $i(x)$ is a convex polygon based at x in $\mathbb{E}$ (possibly the degenerate polygon $\{x\}$). These polygons are isometric to polygons obtained as the intersections of planes with 3-cubes. We wish to estimate a lower bound for the total angle sum about x. It is hard to determine the angle at x in each polygon, but it is much easier if we group polygons together. Here are the details.

Suppose $i(x) = p$ is an interior point of an edge l of $\tilde{L}$. Then each pair of adjacent polygons at $x \in \mathbb{E}$ maps isometrically to the intersection of $i(\mathbb{E})$ with two 3-cells of X which share a common 2-face. This 2-face contains the edge l. Viewing this locally at p we have a plane intersecting a half space in $\mathbb{R}^3$. Thus, these two polygons contribute exactly π to the angle sum at x. Now we shall show that there are at least 6 polygons about x in $\mathbb{E}$. this gives an angle sum of at least 3π at x, contradicting the fact that $\mathbb{E}$ is a flat plane.

One may define the link of p in X as we did for vertices. Note that $Lk(p, X)$ is a piecewise spherical cell complex, made from right angled spherical bigons. We also have a cell structure defined locally at x, and so have a map of links $i_* : Lk(x, \mathbb{E}) \to Lk(p, X)$ whose image does not intersect $Lk(p, \tilde{L})$.

Let v be an endpoint of the edge l. Since $v \in \tilde{L}$ Lemma 6.4 (2) implies that $Lk(v, X)$ is isomorphic to $\Gamma * F$. Define a map

$$pr : \bigcup \{ e \ : \ e \text{ is a 3-cell of } X \text{ which contains } l \} \ \to \ X$$

locally, so that it projects each such 3-cell e onto the face which contains v and is perpendicular to l. Then we have a well defined map

$$Lk(p, X) \setminus Lk(p, \tilde{L}) \;\to\; \Gamma \;\subset\; Lk(v, X)$$

taking a unit vector u based at p to the unit vector $pr_*(u)/|pr_*(u)|$. Composing with i_* we map $Lk(x, \mathbb{E})$ onto a closed circuit in Γ.

Now $\Gamma \to (F * F)$ is a finite cover of the bipartite graph $(\Gamma * \Gamma)$, and so Γ is also bipartite. By Property 1 each of the 36 minimal length closed circuts in $(F * F)$ have connected 5-fold cover preimages in Γ. Thus, the minimal length of a closed circut in Γ is at least 6. Hence, there are at least 3 pairs of adjacent polygons at $x \in \mathbb{E}$ as required.

This is the first reason for our careful choice of covers $\overline{\Delta} \to \Delta$ and $\overline{\Theta^3 \setminus L} \to \Theta^3 \setminus L$. Each lift to Γ of each of the 36 loops $(S^0 * S^0) \subset (F * F)$ needs to be a nontrivial covering.

The case where $i(x) = p$ is a vertex of $\tilde{L}$ is treated similarly. In this case, there are four edges of $\tilde{L}$ containing p. One can still define the projection map pr locally as above ($p = v$ here). The local picture at p is a little more complicated. An angle sum of π will correspond to the intersection of a half plane (satisfying $z \geq 0$) through the origin in $\mathbb{R}^3$ with a collection of four unit cubes based at the origin in the upper half space ($z \geq 0$). The point p maps to the origin and two of the edges of $\tilde{L}$ map to the y-axis.

The hyperplane will typically intersect three of the four cubes, and so the preimage in $\mathbb{E}$ may correspond to a sequence of three polygons about x. However, this still just corresponds to a path of length 2 in the graph $\Gamma \subset Lk(p, X)$, and we still get a local angle sum of at least 3π as above. $\square$

Proof of Theorem 6.1 part (4). The branched covering map $b : Y \to \Theta^3$ restricts to give a local isometry on each 3-cube of Y. Therefore, the composite $f = db$ lifts to give a ϕ-equivariant Morse function

$$\tilde{f} : X \;\to\; \mathbb{R}.$$

It remains to check that the ascending and descending links of $\tilde{f}$ are all 2-spheres.

The local picture of $\tilde{f}$ at a vertex of type I in X is the same as the local picture of d at a vertex of Θ^3, and so the ascending and descending links are all isomorphic to the 3-fold join of 0-spheres.

To understand the local picture of $\tilde{f}$ at a vertex of type II we note that f is defined as the composite of d with the branched covering map b. The link of type II vertices is described in Lemma 6.4 (2). The ascending/descending links of $\tilde{d}$ are octahedra

$$S^0 * S^0 * S^0 \;\subset\; F * F * F$$

These lift to X to become subcomplexes of the links $F * \Gamma$. By Lemma 5.8 (2), each of the loops of the form $S^0 * S^0 \subset F * F$ has connected preimage in Γ. Thus the ascending/descending links of $\tilde{f}$ at type II vertices are 2-spheres of the form $S^0 * S^1$ where the simplicial structure on S^1 is defined as the boundary of a 20-gon.

Note that if the loops $S^0 * S^0 \subset F * F$ had disconnected preimages in Γ then the ascending/descending links of $\tilde{f}$ would be the suspension of a disconnected graph, and so would have nontrivial π_1. This is the second reason for our careful choice of covers $\overline{\Delta} \to \Delta$ and $\overline{\Theta^3 \setminus L} \to \Theta^3 \setminus L$. $\square$

This finishes the proof of Theorem 6.1 $\square$

§7. Questions

There appears to be an interesting interplay between the geometry and topology in this example. On the one hand, one needs a lot of branching to ensure that the universal cover of the cubical complex Y is δ-hyperbolic. On the other hand, branched covering affects the local link structure of these cubical complexes. One may end up with a Morse function whose ascending/descending links are not 0- or 1-connected. In such cases, the kernel H may not be finitely presented. One needs to keep this in mind when considering the following questions.

Question 7.1. Does there exist a hyperbolic group G containing a subgroup H which is finitely presented, of type FP_n, but not of type FP_{n+1} for $n \geq 3$?

If such groups G exist what is their cohomological dimension? How does the minimal cohomological dimension of such G relate to the integer n?

It is tempting to try and generalize the construction in this paper in order to produce examples of hyperbolic groups G containing finitely presented subgroups of type FP_3 but not of type FP_4. One might consider a map from the direct product of four graphs like Θ to S^1, and then take branched coverings over some generically chosen 2-subcomplexes. However, the universal covers of the resulting branched coverings always contain isometrically embeddedd flat planes, and so the groups G are not hyperbolic. M. Bestvina first noticed this phenomenon in Gromov's example.

The basic problem is that there are always embedded flat planes in the universal cover of a finite branched covering of the standard 4-torus $(\mathbb{R}^4/\mathbb{Z}^4)$, when the branching locus consists of generically chosen 2-subtori which are parallel to the coordinate 2-subtori. See [Be] for details. We have tried to remedy this situation by choosing more complicated branching loci, so that the universal covers become δ-hyperbolic. However, the local link structure always changes so that some of the ascending/descending links are no longer 1-connected.

Question 7.2. Does there exist a hyperbolic group G containing a finitely presented subgroup H of type FP_∞ (or of type FP) which is not hyperbolic?

There are many interesting questions concerning the geometry of this example. For example, since the subgroup H is not hyperbolic, it must have an isoperimetric inequality which is at least quadratic. What is the isoperimetric inequality of H?

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