On algebraic and geometric dimensions for groups with torsion

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Abstract We compare various notions of dimension for discrete groups. We exhibit a group that acts with finite stabilizers on an acyclic 2-complex in such a way that the fixed point subcomplex for any non-trivial finite subgroup is contractible, but such that the group does not admit any such action on a contractible 2-complex. This group affords a counterexample to a natural generalization of the Eilenberg-Ganea conjecture.

1. INTRODUCTION

Let G be a discrete group. A G-CW-complex is a CW-complex X equipped with a G-action such that all elements of G map cells of X homeomorphically to cells, and such that whenever an element of G maps a cell to itself it does so via the identity map. A free G-CW complex is one in which the stabilizer of each cell is trivial. A free G-CW-complex X is said to be a model for EG if any of the following equivalent conditions hold:

(a) X/G is an Eilenberg-Mac Lane space $K(G,1)$ for the group G;
(b) X is contractible;
(c) X is a terminal object in the homotopy category of free G-CW-complexes.

Condition (c) means that for any free G-CW-complex Y there is an equivariant map from Y to X, and that any two such maps are equivariantly homotopic. Constructions due to Milnor [25] and Segal [30] show that for every G, there is a model for EG.

It is customary to define the geometric dimension, $\mathrm{gd}G$, of a discrete group G to be the minimal dimension of a model for EG. Similarly, the cohomological dimension, $\mathrm{cd}G$, of such a group is defined to be the projective dimension of $\mathbb{Z}$ over the integral group ring $\mathbb{Z}G$, where $\mathbb{Z}$ is made into a $\mathbb{Z}G$-module by letting each element of G act as the identity. (Both quantities are either positive integers or infinity.) From these definitions it follows that for any subgroup H of G, $\mathrm{gd}H \leq \mathrm{gd}G$ and $\mathrm{cd}H \leq \mathrm{cd}G$. The cellular chain complex for a model for EG is a free resolution for $\mathbb{Z}$ over $\mathbb{Z}G$, and hence for any G, $\mathrm{cd}G \leq \mathrm{gd}G$.

In the 1950's Eilenberg and Ganea showed that for any G, $\mathrm{gd}G \leq \max\{\mathrm{cd}G, 3\}$ [15], and in the 1960's Stallings (for the finitely generated case) and Swan (general case) showed that if $\mathrm{cd}G = 1$, then G is free, and hence $\mathrm{gd}G = 1$ [31,32]. It follows that $\mathrm{gd}G = \mathrm{cd}G$, except that possibly there may exist a group G for which $\mathrm{cd}G = 2$ and $\mathrm{gd}G = 3$. The statement that $\mathrm{cd}G$ and $\mathrm{gd}G$ are always equal has become known as the Eilenberg-Ganea conjecture. The construction used in the proof that $\mathrm{gd}G \leq \max\{\mathrm{cd}G, 3\}$ also show that whenever $\mathrm{gd}G \leq 2$, there is an *acyclic* 2-dimensional free G-CW-complex. Thus the Eilenberg-Ganea conjecture is equivalent to the statement that whenever a group admits a free action on an acyclic 2-complex, it also admits a free action on a contractible 2-complex. At least two families of potential counterexamples have been suggested, the first by M. Bestvina and M. Davis [2], and the second by M. Bestvina and one of the current authors [3].

It is easy to show that if H is a non-trivial finite cyclic group then $\mathrm{cd}H = \infty$. It follows that if G is any group containing torsion, then $\mathrm{gd}G = \mathrm{cd}G = \infty$. This may be viewed as a drawback with the definitions of geometric and cohomological dimension, since for many groups G there are 'naturally occurring' examples of finite-dimensional contractible spaces with G-actions that are almost free, in the sense that all stabilizers are finite. For example, suppose that $\mathcal{G}$ is a Lie group with finitely many components, that $\mathcal{K}$ is a maximal compact subgroup of $\mathcal{G}$, and that G is a discrete subgroup of $\mathcal{G}$. Then the coset space $\mathcal{G}/\mathcal{K}$ is a model for EG if and only if G is torsion-free.

Motivated by these examples, we consider defining a different notion of geometric dimension, using actions with finite stabilizers instead of free actions. Define a proper G-CW-complex to be a G-CW-complex in which all stabilizers are finite (this is equivalent to requiring that G should act properly discontinuously). Say that a proper G-CW-complex X is a model for $\underline{E}G$ if either of the following conditions holds:

(b′) for every finite $H \leq G$, the fixed point set X^H is contractible;
(c′) X is a terminal object in the homotopy category of proper G-CW-complexes.

An easy obstruction-theoretic argument shows that these conditions are equivalent, and a generalization of either Milnor's or Segal's construction establishes the existence of a model for $\underline{E}G$. Note however, that there is no analogue of the condition (a) that characterized models for EG. Since G does not act freely on $\underline{E}G$, it is not possible in general to recover G from the space $\underline{E}G/G$. There are many groups G for which $\underline{E}G/G$ is contractible, and it may be shown that any connected CW-complex is homotopy equivalent to $\underline{E}G/G$ for some discrete group G [20]. The space $\underline{E}G$ plays a rôle in the Baum-Connes conjecture [1]. (See for example [23] for a proof that the G-CW-complex we describe is homotopy equivalent to the metric space used in [1].)

Define $\underline{\mathrm{gd}}G$, the proper geometric dimension of G, to be the minimal dimension of a model for $\underline{E}G$. As for gd, it is easy to see that when $H \leq G$, $\underline{\mathrm{gd}}H \leq \underline{\mathrm{gd}}G$. For torsion-free groups, proper actions are necessarily free, and hence $\underline{\mathrm{gd}}G = \mathrm{gd}G$ whenever G is torsion-free. However, in the case when G is a finite group, a single point will suffice as a model for $\underline{E}G$, and hence in this case $\underline{\mathrm{gd}}G = 0 \neq \mathrm{gd}G = \infty$. In fact it may be shown that $\underline{\mathrm{gd}}G = 0$ if and only if G is finite. As another example, the natural action of the infinite dihedral group on the real line makes the line a 1-dimensional model for $\underline{E}G$, and hence the infinite dihedral group has proper geometric dimension one.

There are a number of possible candidates for an algebraic counterpart to $\underline{\mathrm{gd}}G$, including virtual cohomological dimension, rational cohomological dimension, and relative projective dimension (relative to all finite subgroups). These will be defined and compared below. A number of recent papers have given bounds for $\underline{\mathrm{gd}}G$ in terms of algebraic data [19,27,22,23]. The closest algebraic analogue of $\underline{\mathrm{gd}}G$ involves homological algebra in the category of Bredon coefficient systems for proper G-CW-complexes. This is an abelian category, containing a 'constant object' which we shall denote by $\mathbf{Z}$, and we define $\underline{\mathrm{cd}}G$ to be the projective dimension of $\mathbf{Z}$. It follows easily from the definitions that $\underline{\mathrm{cd}}G = \mathrm{cd}G$ whenever G is torsion-free, and that for any G, $\underline{\mathrm{cd}}G \leq \underline{\mathrm{gd}}G$. Lück has shown that $\underline{\mathrm{gd}}G \leq \max\{3, \underline{\mathrm{cd}}G\}$, a direct analogue of the Eilenberg-Ganea theorem ([21], but see also [23] where an analogous result for certain topological groups is proved). A result of Dunwoody concerning groups of rational cohomological dimension one implies that whenever $\underline{\mathrm{cd}}G = 1$, $\underline{\mathrm{gd}}G = 1$ [14], and so $\underline{\mathrm{gd}}G = \underline{\mathrm{cd}}G$ except that possibly there may exist G for which $\underline{\mathrm{gd}}G = 3$ and $\underline{\mathrm{cd}}G = 2$. Thus one may view the statement that $\underline{\mathrm{cd}}G = \underline{\mathrm{gd}}G$ for all groups G as an extension of the Eilenberg-Ganea conjecture to groups containing torsion.

Just as in the case of $\mathrm{cd}G$, there is a topological interpretation of the conjecture: Lück's argument [21] may be adapted to show that if $\underline{\mathrm{cd}}G = 2$, then G acts properly on an *acyclic* 2-complex, in such a way that the fixed point set for any finite subgroup is *acyclic*.

We exhibit a family of groups G for which $\underline{\mathrm{gd}}G = 3$ but $\underline{\mathrm{cd}}G = 2$. These groups were already known to Bestvina and to Davis. The groups contain finite-index torsion-free subgroups, which are expected to afford counterexamples to the usual Eilenberg-Ganea conjecture [2]. In the words of Mike Davis, our contribution has been to find a question that is answered by these examples. The extra ingredient that makes the extended conjecture accessible is that $\underline{E}G$ contains a subspace $\underline{E}G^{\mathrm{sing}}$, well-defined up to G-homotopy, consisting of the points whose stabilizer is non-trivial. For groups in the family we consider, there is an explicit model X for $\underline{E}G$ such that X^{sing} is acyclic and 2-dimensional, X/G is contractible, and X^{sing}/G is an acyclic 2-complex that cannot be embedded in any contractible 2-complex.

The remainder of the paper is organized as follows. In section 2, we describe the category of Bredon coefficient systems, discuss other notions of algebraic dimension for groups with torsion, and show that they are all bounded above by $\underline{\mathrm{cd}}$. We also give a sketch proof of Lück's theorem relating $\underline{\mathrm{cd}}G$ and $\underline{\mathrm{gd}}G$. In section 3, we describe Davis' construction of contractible simplicial complexes with proper actions of right-angled Coxeter groups G, explain why these complexes are models for $\underline{E}G$, and describe the examples mentioned above. Finally in section 4 we show that there are acyclic 2-complexes that are not contained in any contractible 2-complex. Much of each section consists of a survey of known results.

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2. ALGEBRAIC DIMENSION

We define some of the candidates for algebraic dimension, and discuss their properties. We begin with some information concerning the category of Bredon coefficient systems on proper complexes. For further information on this topic, see [6,21,22].

The orbit category $\mathcal{O}(G)$ is the category whose objects are transitive G-sets G/H, with morphisms from G/H to G/K the set $\mathrm{Map}_G(G/H, G/K)$ of G-maps. Note that the endomorphism monoid of the object G/H is isomorphic to the Weyl group $W(H)$ of H, defined in terms of the normalizer of H by $W(H) = N_G(H)/H$. The category $\mathcal{O}(\mathcal{F}, G)$ is the full subcategory of $\mathcal{O}(G)$ with objects the G/H for H finite. Note that when H is torsion-free, $\mathcal{O}(\mathcal{F}, G)$ has just one object $G/1$, with endomorphism monoid G.

The category of Bredon coefficient systems on proper G-CW-complexes, or category of $\mathbb{Z}\mathcal{O}(\mathcal{F}, G)$-modules (this notation is used in [21,22,23,27]), is the category of contravariant functors from $\mathcal{O}(\mathcal{F}, G)$ to abelian groups. Like any category of functors to abelian groups, this category is an abelian category, in which a sequence is exact if and only if its value on each object G/H is an exact sequence of abelian groups. If X is a proper G-CW-complex then the maps $G/H \mapsto H_i(X^H)$ and $G/H \mapsto C_i(X^H)$ give rise to $\mathbb{Z}\mathcal{O}(\mathcal{F}, G)$-modules, where $C_i(-)$ denotes the ith cellular chain group.

For each finite $K \leq G$, there is a $\mathbb{Z}\mathcal{O}(\mathcal{F},G)$-module $P_{G/K}$ defined by $P_{G/K}(G/H) = \mathbb{Z}\mathrm{Map}_G(G/H,G/K)$, the free abelian group on the set of G-maps from G/H to G/K. The set of morphisms from $P_{G/K}$ to another object $M(-)$ is naturally isomorphic to the abelian group $M(G/K)$. It follows that $\mathrm{Hom}(P_{G/K},-)$ is an exact functor, and so $P_{G/K}$ is projective. Call $P_{G/K}$ the free module of rank 1 based at G/K, and call a direct sum of such modules a free module. It may be shown that any module is a quotient of a free module, and hence any projective module is a direct summand of a free module. The cellular chain groups for a proper G-CW-complex X (together with the cellular chain groups for the fixed point subcomplexes X^H) form free modules, with one copy of $P_{G/K}$ in $C_i(X^{(-)})$ for every orbit of i-cells of type G/K. It follows that the cellular chain complex for a model for $\underline{E}G$ is a free resolution for the module $\mathbf{Z}$, defined on objects by $\mathbf{Z}(G/H) = \mathbb{Z}$ and by the identity map $1_{\mathbb{Z}}$ on each morphism.

In the case when G is torsion-free, the category of $\mathbb{Z}\mathcal{O}(\mathcal{F},G)$-modules is just the usual category of G-modules, and the final assertion of the above paragraph is just the familiar fact that the cellular chain complex for $EG = \underline{E}G$ is a free resolution for $\mathbb{Z}$ over the group ring $\mathbb{Z}G$. The Bredon cohomology of a proper G-CW-complex X with coefficients in a $\mathbb{Z}\mathcal{O}(\mathcal{F},G)$-module $M(-)$ is defined to be the cohomology of the cochain complex $\mathrm{Hom}(C_*(X^{(-)}),M(-))$. In the case when X is a model for $\underline{E}G$, these cohomology groups may be described as $\mathrm{Ext}^*(\mathbf{Z}(-),M(-))$. In the torsion-free case, this specializes to the familiar fact that group cohomology may be defined either as the equivariant cohomology of EG or as Ext groups over $\mathbb{Z}G$.

Theorem 1. *(W. Lück, [21] Thrm 13.19) Let G be a group with $\underline{\mathrm{cd}}G = n$. If $n \geq 3$, there is an n-dimensional model for $\underline{E}G$. If $n = 2$, there is a 2-dimensional proper G-CW-complex X such that X^H is acyclic for every finite subgroup H of G.*

Sketch proof. Let Z be the $(n-1)$-skeleton of any model for $\underline{E}G$. Since $G/H \mapsto C_*(Z^H)$ is a partial free resolution for $\mathbf{Z}$ of length $n-1$, it follows that $H_{n-1}(Z^{(-)})$ is a projective $\mathbb{Z}\mathcal{O}(\mathcal{F},G)$-module, and that $Z^{(-)}$ is $(n-2)$-connected in the sense that for each finite H, Z^H is $(n-2)$-connected. Since any projective is a summand of a free, an Eilenberg Swindle (see for example [8], Prop. 8.2.6) argument shows that there is a free $\mathbb{Z}\mathcal{O}(\mathcal{F},G)$-module $F(-)$ such that $H_{n-1}(Z^{(-)}) \oplus F(-)$ is free. Since Z^H is non-empty for each finite H, we may attach orbits of $(n-1)$-spheres of types $S^{n-1} \times G/H$ to Z to obtain Y, a proper G-CW-complex such that Y^H is $(n-2)$-connected for each finite H and such that $H_{n-1}(Y^{(-)})$ is free. Applying Hurewicz's theorem to each Y^H, it may be seen that proper orbits of n-cells may be attached to Y to obtain X, an n-dimensional proper G-CW-complex such that for each finite H, X^H is acyclic, and is contractible provided that $n \geq 3$. ∎

For the free $\mathbb{Z}\mathcal{O}(\mathcal{F},G)$-module $P_{G/K}$, the $\mathbb{Q}G$-module $\mathbb{Q} \otimes P_{G/K}(G/1)$ is isomorphic to $\mathbb{Q}G/K$, which is projective. It follows that if $P_*(-)$ is any projective resolution of $\mathbf{Z}$, then $\mathbb{Q} \otimes P_*(G/1)$ is a $\mathbb{Q}G$-resolution of $\mathbb{Q} \otimes \mathbf{Z}(G/1) = \mathbb{Q}$ of the same length. Hence $\underline{\mathrm{cd}}G$ provides an upper bound for $\mathrm{cd}_{\mathbb{Q}}G$, the projective dimension of $\mathbb{Q}$ as a $\mathbb{Q}G$-module.

Now suppose that $\underline{\mathrm{cd}}G = 1$. This implies that $\mathrm{cd}_{\mathbb{Q}}G = 1$, and Dunwoody proved that this implies that G acts on a tree T_G with finite stabilizers ([12] IV.3.13 or [14]). For any action of a finite group on a tree, the fixed point set is contractible ([12] I.4, or section 3 below), and so T_G is a 1-dimensional model for $\underline{E}G$. This proves the analogue of the

Stallings-Swan theorem in the context of proper actions, i.e., if $\underline{\mathrm{cd}}G = 1$, then $\underline{\mathrm{gd}}G = 1$.

There are two other algebraic quantities that we will compare to $\underline{\mathrm{gd}}G$, which we will denote by $\mathrm{vcd}G$ and $\mathcal{F}$-$\mathrm{cd}G$. The first of these is the virtual cohomological dimension of G, defined to be equal to $\mathrm{cd}H$ for any finite-index torsion-free subgroup of G. (See [8], VIII, sections 3 and 11 for a proof that this quantity is independent of the choice of finite-index torsion-free subgroup H.) If G contains no torsion-free finite-index subgroups, it is arguable whether $\mathrm{vcd}G$ should take the value ∞, or should be considered to be undefined. We shall take $\mathrm{vcd}G = \infty$ in this case. If H is any torsion-free subgroup of G, then any model for $\underline{E}G$ is also a model for EH. Hence whenever $\mathrm{vcd}G < \infty$, it follows that $\mathrm{vcd}G \leq \underline{\mathrm{gd}}G$.

One drawback with $\mathrm{vcd}G$ is that there are groups G with no torsion-free finite-index subgroups that nevertheless have finite dimensional models for $\underline{E}G$. For example, any countable locally finite group G has $\mathrm{cd}_{\mathbb{Q}}G = 1$, so has a 1-dimensional model for $\underline{E}G$ by the theorem of Dunwoody mentioned in the previous paragraph [14,12]. (A direct construction of a tree for this group can also be made.) A group G with $\underline{\mathrm{gd}}G = 2$, whose finite subgroups have bounded order is constructed in [13], but this is rather overshadowed by the following example due to Schneebeli [29] (neither of these references mentions $\underline{E}G$). Recall that Higman constructed an infinite finitely presented group H with no non-trivial finite quotients. From the construction it is apparent that there is a finite 2-dimensional $K(H,1)$, and that H is acyclic in the sense that $K(H,1)$ has the same integral homology as a point. Now let Q be a free product of two copies of H amalgamated along an infinite cyclic subgroup. This Q has no non-trivial finite quotients, and there is a finite 2-dimensional $K(Q,1)$ which has the integral homology of a 2-sphere. For any prime p, let G be the central extension of $\mathbb{Z}/p$ by Q corresponding to a non-zero element of $H^2(Q;\mathbb{Z}/p) \cong \mathbb{Z}/p$. Any model for EQ is also a model for $\underline{E}G$, with $\mathbb{Z}/p$ acting trivially, so $\underline{\mathrm{gd}}G = 2$, but G contains torsion and has no finite-index subgroups at all.

Finally, we define $\mathcal{F}$-$\mathrm{cd}G$ in terms of homological algebra relative to all finite subgroups of G. For an introduction to relative homological algebra see [24], and for more details concerning $\mathcal{F}$-$\mathrm{cd}G$ see [27]. A G-module is called $\mathcal{F}$-projective if it is a direct summand of a sum of modules induced from finite subgroups, i.e., M is $\mathcal{F}$-projective if there exists N, finite subgroups H_α and H_α-modules M_α such that

$$M \oplus N \cong \bigoplus \mathbb{Z}G \otimes_{\mathbb{Z}H_\alpha} M_\alpha.$$

An exact sequence of G-modules is said to be $\mathcal{F}$-split if it admits an H-splitting for every finite $H \leq G$. The relative cohomological dimension of G, or $\mathcal{F}$-$\mathrm{cd}G$, is defined to be the minimal length of an $\mathcal{F}$-split resolution of $\mathbb{Z}$ by $\mathcal{F}$-projective modules. For any finite $H \leq G$, any model for $\underline{E}G$ admits an H-equivariant contraction (to an H-fixed point). To prove this, write down an explicit contraction for one of the standard models, and use the fact that any two are G-homotopy equivalent. It follows that the cellular chain complex for any model for $\underline{E}G$ is an $\mathcal{F}$-split resolution of $\mathbb{Z}$ by $\mathcal{F}$-projective modules, and hence $\mathcal{F}$-$\mathrm{cd}G \leq \underline{\mathrm{gd}}G$.

Theorem 2. *For any G the following inequalities hold:*

$$\mathrm{cd}_{\mathbb{Q}}G \leq \mathcal{F}\text{-}cdG \leq \underline{\mathrm{cd}}G \leq \underline{\mathrm{gd}}G.$$

If G contains a finite-index torsion-free subgroup then:

$$\mathrm{cd}_{\mathbb{Q}}G \leq \mathrm{vcd}G \leq \mathcal{F}\text{-}cdG.$$

Proof. We have already shown that $\underline{\mathrm{cd}}G \leq \underline{\mathrm{gd}}G$. The fact that $\mathcal{F}$-cd$G \leq \underline{\mathrm{cd}}G$ was proved in [27], so we merely provide a sketch here. If

$$0 \to P_n(-) \to P_{n-1}(-) \to \cdots \to P_0(-) \to \mathbf{Z} \to 0$$

is a projective resolution over $\mathbb{Z}\mathcal{O}(\mathcal{F}, G)$, then $P_*(G/1)$ is a resolution of $\mathbb{Z} = \mathbf{Z}(G/1)$ by $\mathcal{F}$-projective G-modules. To show that this sequence is H-split for any finite H, it suffices to show that it becomes G-split after tensoring with the permutation module $\mathbb{Z}G/H$. For this one introduced tensor products of $\mathbb{Z}\mathcal{O}(\mathcal{F}, G)$-modules. The tensor product over $\mathbb{Z}$ of $M(-)$ and $N(-)$ may be defined by $(M \otimes N)(G/H) = M(G/H) \otimes N(G/H)$, tensoring with a projective is exact, and the tensor product of two projectives is projective [27]. (In terms of topology, the tensor product corresponds to the direct product of G-CW-complexes.) Now for any finite $H \leq G$, the tensor product $P_*(-) \otimes P_{G/H}(-)$ is a projective resolution for the projective module $P_{G/H}(-)$, so is split. Evaluating this splitting on the object $G/1$ gives the required G-splitting.

$\mathrm{cd}_{\mathbb{Q}}G \leq \mathcal{F}$-cd$G$: For H finite, every $\mathbb{Q}H$-module is projective. It follows that whenever M is a $\mathcal{F}$-projective G-module, $\mathbb{Q}G \otimes_{\mathbb{Z}G} M$ is a projective $\mathbb{Q}G$-module. Since $\mathbb{Q}G \otimes_{\mathbb{Z}G} -$ is exact, it follows that any $\mathcal{F}$-projective resolution of $\mathbb{Z}$ gives rise to a projective resolution of $\mathbb{Q}$ over $\mathbb{Q}G$ of the same length.

$\mathrm{cd}_{\mathbb{Q}}G \leq \mathrm{vcd}G$: This result is well-known so we give only a sketch. If H is a finite-index subgroup of G, then $\mathrm{cd}_{\mathbb{Q}}H = \mathrm{cd}_{\mathbb{Q}}G$ by the rational version of the theorem of Serre that is used to prove that vcd$G$ is well-defined ([8] contains the integral version, or see [4] for the general case). So $\mathrm{cd}_{\mathbb{Q}}G = \mathrm{cd}_{\mathbb{Q}}H \leq \mathrm{cd}H = \mathrm{vcd}G$.

$\mathrm{vcd}G \leq \mathcal{F}$-cd$G$: Suppose that there is an $\mathcal{F}$-split $\mathcal{F}$-projective resolution of $\mathbb{Z}$ of length n, and let P_* be a (possibly longer) resolution that is $\mathbb{Z}$-free as well (for example the cellular chain complex of a model for $\underline{E}G$). Define a module Q_n as

$$Q_n = \ker\left(P_{n-1} \to P_{n-2}\right).$$

It may be shown that Q_n is $\mathcal{F}$-projective and that the shorter exact sequence given below is $\mathcal{F}$-split:

$$0 \to Q_n \to P_{n-1} \to P_{n-2} \to \cdots \to P_0 \to \mathbb{Z} \to 0.$$

Hence there is a $\mathbb{Z}$-free $\mathcal{F}$-split $\mathcal{F}$-projective resolution of length n. If H is a torsion-free subgroup of G, any $\mathcal{F}$-projective G-module restricts to H as a summand of a module induced up from the trivial subgroup. Hence any $\mathbb{Z}$-free resolution by $\mathcal{F}$-projectives restricts to H as a resolution by $\mathbb{Z}H$-projective modules, and so $\mathrm{vcd}G \leq \mathrm{cd}H \leq \mathcal{F}$-cd$G$. ∎

We know of no group G for which $\mathcal{F}$-cd$G$ and $\underline{\mathrm{cd}}G$ are not equal, nor any virtually torsion-free group for which $\mathrm{vcd}G \neq \underline{\mathrm{cd}}G$. Bestvina constructed groups G for which $\mathrm{vcd}G = 3$ but $\mathrm{cd}_{\mathbb{Q}}G = 2$ [2], and taking direct products of these gives groups for which $\mathrm{cd}_{\mathbb{Q}}G = 2n$ and $\mathrm{vcd}G = 3n$ for any n. Some further related examples occur in [13], including a group

G for which $\mathrm{cd}_F G = 2$ a field F of any characteristic, whereas $\mathrm{cd}G = 3$. (Note that such a G cannot be of type FP, whereas Bestvina's groups are.)

On the other hand, there is no known upper bound for $\underline{\mathrm{gd}}G$ in terms of $\mathcal{F}$-cd$G$ or vcd$G$ alone. The bound given in [27] involves $\mathcal{F}$-cd$G$ and the maximal length l of a chain of finite subgroups in G. There is one easily obtainable bound in terms of vcd$G$. If there is an m-dimensional model X for EH, there is an action of the wreath product $W = H \wr \Sigma_n$ upon X^n which makes it a model for $\underline{E}W$. (This construction may be mimicked algebraically too but is slightly more complicated [8,4].) Any G containing H as an index n subgroup may be embedded in W, and hence $\underline{\mathrm{cd}}G \leq n.\mathrm{vcd}G$ whenever G contains a torsion-free subgroup of index n.

3. RIGHT-ANGLED COXETER GROUPS

In [9], M. Davis introduced, for any Coxeter group G, a contractible simplicial complex with a proper G-action. (We shall only need to consider the case when G is a right-angled Coxeter group, which will be explained below.) Davis' complex turns out to be a model for $\underline{E}G$. This could be proved purely algebraically, but the easiest way to check it is to use the fact that the complex admits a CAT(0)-metric structure, which was proved by Gromov [17,p. 132] in the right-angled case, and by Moussong in the general case [26,11]. Recall that a CAT(0)-metric space is a complete metric space, containing a geodesic arc between each pair of points, such that for every triple of points, any geodesic triangle between them is no wider than the triangle in the Euclidean plane with the same edge lengths. (The case when one vertex of the triangle lies on a geodesic between the other two implies that geodesics are unique.) The following proposition shows that a proper G-CW-complex with a compatible CAT(0)-metric is necessarily a model for $\underline{E}G$.

Proposition 3. *Any CAT(0)-space is contractible. If a finite group acts by isometries on a CAT(0)-space then the fixed points form a contractible subspace.*

Sketch proof. Let X be a CAT(0)-space with metric d and let H act on X. For any $x \in X$, define a map $c_x : X \times I \to X$ by defining $c_x(y,t)$ to be the point at distance $t.d(x,y)$ from x along the geodesic arc from x to y. This is a contraction of X to x. If x and y are fixed by H, so is the geodesic arc between them, so provided that $X^H \neq \emptyset$, c_x restricts to a contraction of X^H. It remains to show that $X^H \neq \emptyset$.

Given any point $x_n \in X$, choose y_n in the H-orbit Hx_n to maximize $d(x_n, y_n)$, and define x_{n+1} to be the midpoint of the geodesic arc between x_n and y_n. For any choice of x_0, this gives rise to a Cauchy sequence (x_n) which converges to an H-invariant point. (In fact, elementary calculations with Euclidean simplices show that $d(x_{n+2}, x_{n+1}) \leq (\sqrt{3}/2)d(x_{n+1}, x_n)$ and that for any $h, h' \in H$, $d(x_{n+1}, hx_{n+1}) \leq (1/\sqrt{2})d(x_n, h'.x_n)$.) ∎

There is another way to find an H-fixed point: for any bounded set B in a CAT(0) space, it can be shown that there is a unique ball of minimal radius containing that set. The centre of B is by definition the centre of this ball [7]. The centre of an H-orbit is a point fixed by H.

A *right-angled Coxeter system* consists of a group G (a *right-angled Coxeter group*) together with a set V of involutions that generate G, subject only to the relations that certain pairs of the generators commute. We shall assume that V is finite. The right-angled Coxeter system is determined by the graph with vertex set V and edges those pairs that commute, or by the flag complex whose n-simplices are the commuting $(n+1)$-tuples of elements of V, which we shall call $K(G,V)$. (A *flag complex* is a simplicial complex that contains a simplex with vertex set W whenever its edge set contains the complete graph on W.) Conversely, any flag complex determines a right-angled Coxeter system with generators the vertices of the complex. Define a *special subgroup* to be a subgroup of G that is generated by a (possibly empty) subset of V.

Given a right-angled Coxeter system (G,V), define a simplicial complex $D(G,V)$ as the realization of the poset of cosets of finite special subgroups of G. The action of G on cosets induces a proper action of G on $D(G,V)$. The complex was introduced by M. Davis, who proved that $D(G,V)$ is contractible [9], and defined a similar complex for an arbitrary Coxeter system. As remarked above, it may be shown that $D(G,V)$ is a model for $\underline{E}G$ by showing that $D(G,V)$ admits a CAT(0)-metric [17,p. 132].

If $W \subseteq V$ generates a finite subgroup of G, then $\langle W\rangle$ is isomorphic to a direct product of n copies of the cyclic group $\mathbb{Z}/2$, where $2^n = |W|$, and the realization of the subposet consisting of all cosets contained in $g\langle W\rangle$ may be identified with the barycentric subdivision of an n-cube, upon which its stabilizer, $g\langle W\rangle g^{-1}$, acts as reflections in hyperplanes through its centre parallel to its $(n-1)$-faces. The original simplicial structure may be recovered from this cubical structure as its barycentric subdivision. A geodesic metric on $D(G,V)$ may be defined by setting each such cube to be isometric to a standard Euclidean n-cube of fixed side length. This defines a metric locally, and hence defines a length function on paths. The distance between any two points of $D(G,V)$ is now defined to be the length of a shortest path between them.

The complex $D(G,V)$ is piecewise-Euclidean cubical, and the link of every vertex of a cube is isomorphic to $K(G,V)$. This is a flag complex, and hence the cubical complex is locally CAT(0) (which means that the CAT(0) condition holds for all sufficiently small triangles) by a lemma due to Gromov [17,10]. To see that $D(G,V)$ is simply connected, compare its cubical 2-skeleton D to C, the universal cover of the presentation 2-complex of G (with the Coxeter presentation). Both C and D have vertex set G. For each $v \in V$ and $g \in G$, there is a 2-sphere in C joining the vertices g and gv (consisting of two lifts of the edge labeled by v and two lifts of the 2-cell corresponding to the relation $v^2 = e$), whereas in D there is a single 1-simplex. Collapsing these 2-spheres to edges gives rise to a complex C' with the same 1-skeleton as D and the same fundamental group as C. The difference between C' and D is that for each $g \in G$ and $v, w \in V$ such that $vw = wv$, there are four squares around the cycle (g, gv, gvw, gw) in C' and only one in D. Identifying these four squares does not change the fundamental group, and hence $\pi_1(D) \cong \pi_1(C') \cong \pi_1(C) = \{1\}$. Since $D(G,V)$ is simply-connected and locally CAT(0), it follows that $D(G,V)$ is CAT(0) [7,10].

For example, in the case when $K(G,V)$ is the boundary of a square, G is a direct product of two infinite dihedral groups. In this case $D(G,V)$ is isometric to a tessellation of the Euclidean plane by squares. This may be compared with the case when $K(G,V)$ is the boundary of a pentagon. In this case G is isomorphic to the group generated by reflections in the sides of a right-angled pentagon in the hyperbolic plane, and $D(G,V)$ is equivariantly homeomorphic to the hyperbolic plane. However, the metric on $D(G,V)$ is not the standard Riemannian metric on the hyperbolic plane, because $D(G,V)$ is made out of Euclidean squares. Five squares meet at every vertex, and so the negative curvature is concentrated at the vertices and the metric is not Riemannian.

If $K(G,V)$ contains a full subcomplex isomorphic to the boundary of a square, it follows that $D(G,V)$ contains copies of the Euclidean plane, and that G contains copies of $\mathbb{Z}\times\mathbb{Z}$. Hence in this case, G cannot be word hyperbolic. Conversely, it may be shown that G is word hyperbolic whenever $K(G,V)$ does not contain any squares as full subcomplexes (this is Gromov's 'no squares condition' — Moussong has described a similar condition in the non right-angled case [17,26,10]).

Proposition 4. *The quotient of the pair $(D(G,V), D(G,V)^{\mathrm{sing}})$ by G is homeomorphic to the pair $(CK(G,V), K(G,V))$, where CX denotes the cone on X. If $K(G,V)$ is acyclic, then so is $D(G,V)^{\mathrm{sing}}$. If $K(G,V)$ is an acyclic 2-complex that is not homotopy equivalent to a subcomplex of a contractible 2-complex, then $\underline{\mathrm{cd}}G = 2 < \underline{\mathrm{gd}}G = 3$.*

Proof. For this proposition it is more convenient to use the simplicial structure on $D(G,V)$. A simplex of $D(G,V)$ is in a free G-orbit if and only if it contains a vertex in a free orbit. Hence $D(G,V)^{\mathrm{sing}}$ is the realization of the poset of cosets of non-trivial finite special subgroups of G. Factoring out by the action of G identifies all the cosets of a given subgroup, so $D(G,V)^{\mathrm{sing}}/G$ is isomorphic to the realization of the poset of non-trivial finite special subgroups of G, or the barycentric subdivision of $K(G,V)$. Also $D(G,V)/G$ is isomorphic to the realization of the poset of all finite special subgroups of G, which is isomorphic to the cone on the barycentric subdivision of $K(G,V)$.

$D(G,V)$ contains one free G-orbit of vertices, the cosets $g.\{e\}$ of the trivial subgroup. There are no edges between these vertices, and so $D(G,V) - G.\{e\}$ retracts onto $D(G,V)^{\mathrm{sing}}$. Hence $D(G,V)$ can be built from $D(G,V)^{\mathrm{sing}}$ by attaching a free G-orbit of copies of the cone on the link of $\{e\}$. This link is homeomorphic to $K(G,V)$, as it is the realization of the poset of cosets of special subgroups strictly containing $\{e\}$. Coning off an acyclic subspace does not change homology, so when $K(G,V)$ is acyclic, $D(G,V)^{\mathrm{sing}}$ has the same homology as $D(G,V)$ and is therefore acyclic.

For any group H, let X and Y be two models for $\underline{E}H$. The universal property tells us that there are equivariant maps from X^{sing} to Y and that any two such maps are equivariantly homotopic. The image of X^{sing} under such a map or homotopy must be contained in Y^{sing}, and it follows that X^{sing} and Y^{sing} must be equivariantly homotopy equivalent. Also X^{sing}/H and Y^{sing}/H are homotopy equivalent.

If $K(G,V)$ is an acyclic 2-complex, the chain complex for $D(G,V)^{\mathrm{sing}}$ is a free resolution of length two for $\mathbf{Z}$ over the Bredon category, showing that $\underline{\mathrm{cd}}G \leq 2$. If X is any model for $\underline{E}G$, then X/G is contractible and contains the subspace X^{sing}/G which is homotopy equivalent to $D(G,V)^{\mathrm{sing}}/G$. Since no 2-complex homotopy equivalent to $K(G,V)$ can be embedded in a contractible 2-complex, it must be the case that X is at least 3-dimensional.
∎

4. ACYCLIC 2-COMPLEXES

In this section, we give examples of acyclic 2-complexes that are known not to embed in any contractible 2-complex, and sketch the proof of this fact.

Proposition 5. *Let K be an acyclic 2-complex with non-trivial finite fundamental group. Then K does not embed as a subcomplex of a contractible 2-complex.*

Proof. We argue by contradiction. Suppose that K embeds as a subcomplex of a contractible 2-complex L. Then there is a subcomplex $K_1 \subset L$ consisting of K together with finitely many extra cells such that the inclusion of K in K_1 is not injective on fundamental groups. For example, we can take K_1 to be the union of K and the image of a null homotopy in L of a non-trivial element of $\pi_1(K)$.

Now, since K_1 is a subcomplex of the contractible 2-complex L, we conclude that $H_2(K_1;\mathbb{Z}) = 0$ and that $H_1(K_1;\mathbb{Z})$ is finite rank free abelian. We kill off these free generators of H_1 by attaching 2-cells to K_1 to obtain an acyclic 2-complex K_2 which contains K. Since $H_1(K;\mathbb{Z}) = 0$ and $H_2(K_2;\mathbb{Z}) = 0$, it follows that $H_2(K_2, K;\mathbb{Z}) = 0$. By collapsing a maximal forest in $K_2 - K$ we see that, up to homotopy, K_2 is obtained from K by attaching r 1-cells and r 2-cells. We may think of the r 2-cells as giving a system of r equations in r unknowns over $\pi_1(K)$, and the fact that $H_2(K_2, K;\mathbb{Z}) = 0$ ensures that this is a linearly independent system [18]. Since $\pi_1(K)$ is finite, a theorem of Gerstenhaber-Rothaus [16] implies that the inclusion induced map $\pi_1(K) \to \pi_1(K_2)$ is injective. But this contradicts the construction of K_2. ∎

Remark. The condition that the fundamental group of K be finite is only used to deduce that, for any 2-complex M containing K such that $H_2(M, K;\mathbb{Z}) = 0$, the map $\pi_1(K) \to \pi_1(M)$ is injective. This property is also known to hold for many other groups, for example: any group with a finite-dimensional faithful unitary representation [16], any locally residually finite group [28], or any locally indicable group [18] (but note that these last examples cannot be the fundamental groups of acyclic complexes). In fact, no group is known for which this does not hold [18]. As well as [16,18], we recommend [5] as a reference on this topic.

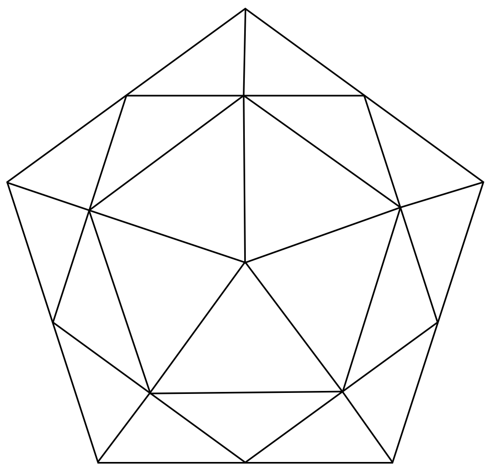

Figure 1.

Finally, we describe a 2-complex satisfying the conditions of Propostion 5. Take a complete graph on 5 vertices $v_1, \ldots, v_5$. In the alternating group A_5, there are two conjugacy classes of elements of order 5, and each element of order 5 is conjugate to its inverse. Fixing one of these conjugacy classes gives a choice of 6 inverse pairs of 5-cycles. Use these to attach 6 pentagonal 2-cells. This gives a 2-complex whose fundamental group is the binary icosahedral group of order 120. This 2-complex may also be obtained by identifying opposite faces of a dodecahedron with a twist of $\pi/5$. From this description it is easily seen to be a 2-spine of the Poincaré homology 3-sphere and hence to be acyclic (see for example [33, p. 34]).

The easiest way to make a flag complex homeomorphic to this 2-complex is to take its barycentric subdivision, a complex consisting of 60 triangles. Alternatively take the barycentric subdivision of the 1-skeleton and subdivide each pentagon as shown in figure 1. This complex satisfies the 'no squares condition', and so as remarked in Section 3, the corresponding Coxeter group is word hyperbolic.

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