HYPERBOLIC HYDRA

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Abstract. We give examples of hyperbolic groups with finite–rank free subgroups of huge (Ackermannian) distortion.

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1. Introduction

1.1. **Our result.** Hyperbolic groups are algorithmically tractable (their word and conjugacy problems are straight–forward) and are characterised by a tree–like property that geodesic triangles in their Cayley graphs are close to tripods [5, 8]. The purpose of this article is to show that none–the–less some harbour extreme wildness within their subgroups—their finite–rank free subgroups, even. We prove (the terminology is explained below):

Theorem 1.1. *There are hyperbolic groups Γ_k for all $k \geq 1$ with free rank–$(k+18)$ subgroups Λ_k whose distortion satisfies $\mathrm{Dist}_{\Lambda_k}^{\Gamma_k} \succeq A_k$—that is, grows at least like the k–th of Ackermann's functions.*

A *distortion function* Dist_H^G measures the degree to which a subgroup $H \leq G$ folds in on itself within G by comparing the intrinsic word metric on H with the extrinsic word metric inherited from G. Suppose S and T are finite generating sets for G and H, respectively. Then

$$\mathrm{Dist}_H^G(n) \ := \ \max\{\, d_T(1,g) \ | \ g \in H \text{ with } d_S(1,g) \leq n \,\}.$$

Up to the following equivalence, capturing qualitative agreement of growth rates, Dist_H^G does not depend on S and T. For $f, g : \mathbb{N} \to \mathbb{N}$, we write $f \preceq g$ when there exists $C > 0$ such that $f(n) \leq Cg(Cn + C) + Cn + C$ for all n. Define $f \simeq g$ when $f \preceq g$ and $g \preceq f$.

Ackermann's $A_k : \mathbb{N} \to \mathbb{N}$ are a family of fast–growing functions defined recursively:

$$\begin{aligned} A_0(n) &= n + 2 \text{ for } n \geq 0, \\ A_k(0) &= \begin{cases} 0 & \text{for } k = 1 \\ 1 & \text{for } k \geq 2, \end{cases} \\ \text{and} \quad A_{k+1}(n+1) &= A_k(A_{k+1}(n)) \text{ for } k, n \geq 0. \end{aligned}$$

In particular, $A_1(n) = 2n$, $A_2(n) = 2^n$ and $A_3(n)$ is the n–fold iterated power of 2. They are representatives of the successive levels of the Grzegorczyk hierarchy of primitive recursive functions—see, for example, [15].

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1.2. **The organisation of this article and an outline of our approach.** Our groups Γ_k are elaborations of the *hydra groups*

$$G_k \ = \ \langle\, a_1, \ldots, a_k, t \ \mid \ t^{-1} a_1 t = a_1, \ t^{-1} a_i t = a_i a_{i-1} \ (\forall i > 1) \,\rangle$$

explored by the second and third authors in [6]. These G_k are CAT(0), free–by–cyclic, biautomatic, and can be presented with only one relator, and yet the subgroups $H_k := \langle a_1 t, \ldots, a_k t\rangle$ are free of rank k and their distortion grows like the k–th of Ackermann's functions: $\mathrm{Dist}_{H_k}^{G_k} \simeq A_k$.

This extreme distortion stems from a phenomenon which can be described as a re–imagining of Hercules' battle with the Lernaean Hydra. A *hydra* is a positive word w on the alphabet $a_1, a_2, \ldots$. Hercules removes the first letter and then the creature regenerates in that each remaining a_i with $i > 1$ becomes $a_i a_{i-1}$. (Each remaining a_1 is unaffected.) This repeats and Hercules triumphs when the hydra is reduced to the empty word ε. The number of steps is denoted $\mathcal{H}(w)$. (Each step encompasses the removal of the first letter and then regeneration.) For example, $\mathcal{H}({a_2}^3) = 7$:

$${a_2}^3 \ \to \ (a_2 a_1)^2 \ \to \ a_1 a_2 {a_1}^2 \ \to \ a_2 {a_1}^3 \ \to \ {a_1}^3 \ \to \ {a_1}^2 \ \to \ a_1 \ \to \ \varepsilon.$$

In [6] it is shown that Hercules will be victorious whatever hydra he faces, but the number of strikes it takes can be huge: the functions $\mathcal{H}_k$, defined by $\mathcal{H}_k(n) = \mathcal{H}({a_k}^n)$, grow like Ackermann's functions: $\mathcal{H}_k \simeq A_k$.

The group G_k is not hyperbolic because it has the subgroup $\langle a_1, t\rangle \cong \mathbb{Z}^2$. We obtain Γ_k by combining G_k with another free–by–cyclic group, which is hyperbolic, in such a way that the hydra phenomenon persists in Γ_k, but the troublesome "Euclidean" relations $t^{-1} a_1 t = a_1$ are replaced by something "hyperbolic."

In Section 2 we will give two presentations $\mathcal{P}_k$ and $\mathcal{Q}_k$ for Γ_k and will prove they are equivalent. $\mathcal{P}_k$ is well suited to proving hyperbolicity: the associated Cayley 2–complex will be shown in Section 3 to contan no isometrically embedded copies of $\mathbb{R}^2$ and so is hyperbolic by the Flat Plane Theorem. $\mathcal{Q}_k$ places Γ_k in a class of free–by–cyclic groups which we show in Section 4 (for $k \geq 2$) contain free subgroups of rank $k+18$ and distortion $\geq A_k$. (In the case $k = 1$, Theorem 1.1 is elementary: take Γ_1 to be a free group and Λ_1 to be Γ_1.)

1.3. **Background.** Other heavily distorted free subgroups of hyperbolic groups have been exhibited by Mitra [12]: for all k, he gives an example with a free subgroup of distortion like a k–fold iterated exponential function and, more extreme, an example where the number of iterations grows like $\log n$. Barnard, the first author and Dani developed Mitra's constructions into more explicit examples that are also CAT(−1) [3]. We are not aware of any example of a hyperbolic group with a finite–rank free subgroup of distortion exceeding that of our examples. Indeed, we do not know of a hyperbolic group with a *finitely presented* subgroup of greater distortion. The Rips construction, applied to a finitely presentable group with unsolvable word problem yields a hyperbolic (in fact, $C'(1/6)$ small–cancellation) group G with a finitely generated subgroup N such that Dist_N^G is not bounded above by any recursive function, but these N are not finitely presentable. (See [1, §3.4], [7, Corollary 8.2], [9, §3, 3.K_3''] and [14].)

Whilst we will not call on it in this paper (as we will give the translation between the presentations $\mathcal{P}_k$ and $\mathcal{Q}_k$ explicitly), a result that lies behind how we came to our examples is that if a 2–complex admits an S^1–valued Morse function all of whose ascending and descending links are trees, then its fundamental group is free–by–cyclic [2]. [The ascending link for our examples is visible in Figure 2 as the subgraph made up of all edges connecting pairs of negative vertices. The descending is that made up of all edges connecting pairs of positive vertices. Both are trees.]

1.4. **Towards an upper bound on distortion.** It seems likely that $\mathrm{Dist}_{\Lambda_k}^{\Gamma_k} \simeq A_k$, but we do not offer a proof that $\mathrm{Dist}_{\Lambda_k}^{\Gamma_k} \preceq A_k$. The proof that $\mathrm{Dist}_{H_k}^{G_k} \preceq A_k$ in [6] may guide a proof that $\mathrm{Dist}_{\Lambda_k}^{\Gamma_k} \preceq A_k$, but that proof is technical and how to carry it over to $\mathrm{Dist}_{\Lambda_k}^{\Gamma_k}$ is not readily apparent. We are content to present here just the lower bound, which we believe is the more significant.

1.5. **Height and quasiconvexity.** A finitely generated subgroup H of a finitely generated group G is *quasiconvex* when $\mathrm{Dist}_H^G(n) \leq Cn$ for some constant C. An infinite subgroup H of a group G has *infinite height* when, for all n, there exist $g_1, \ldots, g_n$ such that $\bigcap_{i=1}^n g_i^{-1} H g_i$ is infinite and $Hg_i \neq Hg_j$ for all $i \neq j$.

As $\Lambda_k \leq \Gamma_k$, for $k \geq 2$, are new examples of non–quasiconvex finitely presented subgroups of hyperbolic groups, they are test cases for the question attributed to Swarup in [13]: if a finitely presented subgroup H of a hyperbolic group G has *finite height*, is H quasiconvex in G? (We thank Ilya Kapovich for drawing our attention to this.)

Our $\Lambda_k \leq \Gamma_k$ do not resolve Swarup's question as they have infinite height for all $k \geq 1$. We explain this using the notation of Section 4. It follows from Proposition 4.8 that $t^i \in \Lambda_k$ if and only if $i = 0$. So $\Lambda_k t^i \neq \Lambda_k t^j$ for all $i \neq j$. And $\bigcap_{i=1}^\infty t^{-i} \Lambda_k t^i$ is infinite since the rank–l free group $\langle b_1, \ldots, b_l \rangle$ is a subgroup of $t^{-i} \Lambda_k t^i$ for all i.

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2. Our examples

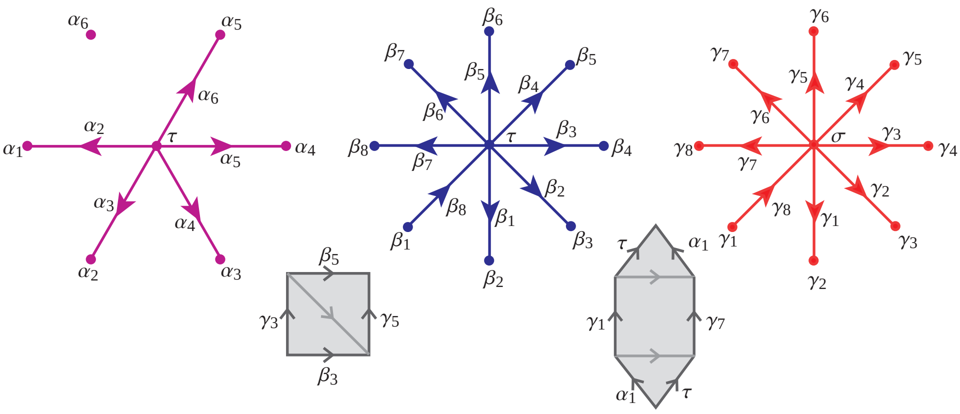

Figure 1. The defining relations of the presentation P_6 for Γ_6 displayed as three LOTs and two 2–cells.

2.1. **A CAT(0) presentation for Γ_k.** This presentation P_k is well suited to establishing hyperbolicity (see Section 3):

generators: $\alpha_1, \ldots, \alpha_k,\ \beta_1, \ldots, \beta_8,\ \gamma_1, \ldots, \gamma_8,\ \sigma, \tau,$

relations:
$$\begin{aligned} {\alpha_i}^{-1}\tau\alpha_i &= \alpha_{i-1} \ (1 < i \le k), \\ {\beta_i}^{-1}\tau\beta_i &= \beta_{i+1} \ (1 \le i \le 7), \quad \beta_8\tau{\beta_8}^{-1} = \beta_1, \\ {\gamma_i}^{-1}\sigma\gamma_i &= \gamma_{i+1} \ (1 \le i \le 7), \quad \gamma_8\sigma{\gamma_8}^{-1} = \gamma_1, \\ \gamma_3\beta_5 &= \beta_3\gamma_5, \quad \alpha_1\gamma_1\tau = \tau\gamma_7\alpha_1. \end{aligned}$$

It is convenient to encode P_k as shown in Figure 1 (which displays the case $k = 6$). Each edge in the three labelled oriented trees (*LOT*s—see [10]) encodes a commutator relation—an edge labelled y from a vertex labelled x to a vertex labelled z corresponds to a relation $y^{-1}xy = z$. The square and hexagonal 2–cells represent the remaining two relations, $\gamma_3\beta_5 = \beta_3\gamma_5$ and $\alpha_1\gamma_1\tau = \tau\gamma_7\alpha_1$.

If one removes the α_i and all the relations in which they appear from P_k, then one essentially gets groups studied by Mecham & Muckerjee in [11]. These, in turn, are built from two copies of groups studied by Barnard and the first author in [2].

2.2. **A presentation of Γ_k as a free–by–cyclic group.** This presentation Q_k has:

generators: $a_0, \ldots, a_k,\ b_1, \ldots, b_8,\ c_1, \ldots, c_8,\ d,\ t,$

relations:
$$\begin{aligned} t^{-1}a_it &= \theta(a_i) && 0 \le i \le k, & t^{-1}c_it &= d\,\psi(c_5)\,\psi(c_i)\,\psi(c_5)^{-1}\,d^{-1} && (1 \le i \le 8), \\ t^{-1}b_it &= \phi(b_i) && 1 \le i \le 8, & t^{-1}dt &= \phi^2(b_5)^{-1}\,d\,\psi(c_5{c_3}^{-1})\,\phi(b_3), \end{aligned}$$

where θ, ϕ and ψ are defined by
$$\begin{aligned} \theta(a_i) &= \begin{cases} ua_1v & i = 0, \\ a_0 & i = 1, \\ a_ia_{i-1} & 1 < i \le k, \end{cases} \\ \phi(b_i) &= (b_i \cdots b_7)\,{b_1}^{-1}b_8 \qquad (1 \le i \le 8), \\ \psi(c_i) &= (c_i \cdots c_8)\,{c_1}^{-1}c_8 \qquad (1 \le i \le 8), \end{aligned}$$

and
$$\begin{aligned} u &= t^{-k}\,{c_7}^{-1}\,t\,d\,\psi(c_5)\,t^{k-1}, \\ v &= t^{-(k-1)}\,\psi(c_5)^{-1}\,d^{-1}\,t^{-1}\,c_1\,t^k. \end{aligned}$$

Lemma 2.1. *Q_k presents a free–by–cyclic group*
$$F(a_0, a_1, \ldots, a_k,\ b_1, \ldots, b_8,\ c_1, \ldots, c_8,\ d) \rtimes \mathbb{Z}$$
where the $\mathbb{Z}$–factor is $\langle t \rangle$ and t acts as an automorphism.

Proof. First, note:

(i) u and v represent elements of the subgroup $\langle b_1, \ldots, b_8,\ c_1, \ldots, c_8,\ d\rangle$, and
(ii) ϕ and ψ define automorphisms of $F(b_1, \ldots, b_8)$ and $F(c_1, \ldots, c_8)$, respectively, as would θ for $F(a_0, \ldots, a_k)$ were $\theta(a_0)$ equal to a_1 rather than ua_1v.

The action of t by conjugation on
$$F(a_0, a_1, \ldots, a_k,\ b_1, \ldots, b_8,\ c_1, \ldots, c_8,\ d)$$

apparent in the presentation Q_k is an automorphism because, as we will explain, the following is a sequence of free bases:

$$
\begin{aligned}
&(a_0, a_1, \ldots, a_k,\ b_1, \ldots, b_8,\ c_1, \ldots, c_8,\ d)\\
&\quad\stackrel{(1)}{\to}\ (a_1, t^{-1}a_1t, \ldots, t^{-1}a_kt,\ b_1, \ldots, b_8,\ c_1, \ldots, c_8,\ d)\\
&\quad\stackrel{(2)}{\to}\ (t^{-1}a_0t, t^{-1}a_1t, \ldots, t^{-1}a_kt,\ b_1, \ldots, b_8,\ c_1, \ldots, c_8,\ d)\\
&\quad\stackrel{(3)}{\to}\ (t^{-1}a_0t, t^{-1}a_1t, \ldots, t^{-1}a_kt,\ t^{-1}b_1t, \ldots, t^{-1}b_8t,\ \psi(c_1), \ldots, \psi(c_8),\ d)\\
&\quad\stackrel{(4)}{\to}\ (t^{-1}a_0t, t^{-1}a_1t, \ldots, t^{-1}a_kt,\ t^{-1}b_1t, \ldots, t^{-1}b_8t,\ \psi(c_1), \ldots, \psi(c_8),\ d\,\psi(c_5{c_3}^{-1}))\\
&\quad\stackrel{(5)}{\to}\ (t^{-1}a_0t, t^{-1}a_1t, \ldots, t^{-1}a_kt,\ t^{-1}b_1t, \ldots, t^{-1}b_8t,\ t^{-1}c_1t, \ldots, t^{-1}c_8t,\ d\,\psi(c_5{c_3}^{-1}))\\
&\quad\stackrel{(6)}{\to}\ (t^{-1}a_0t, t^{-1}a_1t, \ldots, t^{-1}a_kt,\ t^{-1}b_1t, \ldots, t^{-1}b_8t,\ t^{-1}c_1t, \ldots, t^{-1}c_8t,\ t^{-1}dt).
\end{aligned}
$$

This is because (1) $a_1, t^{-1}a_1t, \ldots, t^{-1}a_kt$ is a free basis for $F(a_0, \ldots, a_k)$ as per (ii) above; (2) $t^{-1}a_0t = ua_1v$, which is equivalent via transvections to a_1 by (i); (3) follows from (ii); (4) is via transvections; (5) conjugation by $\psi(c_5)^{-1}\, d^{-1} = \psi(c_5)^{-1}\, \psi(c_5{c_3}^{-1})\, \psi(c_5{c_3}^{-1})^{-1}\, d^{-1}$ is first conjugation by $\psi(c_5)^{-1}\, \psi(c_5{c_3}^{-1})$, which is an automorphism of $F(c_1, \ldots, c_8)$, and then by $\psi(c_5{c_3}^{-1})^{-1}\, d^{-1}$; and (6) is via transvections as $t^{-1}b_1t, \ldots, t^{-1}b_8t$ are a free basis for $F(b_1, \ldots, b_8)$ and $\phi^2(b_5)^{-1}, \phi(b^3) \in F(b_1, \ldots, b_8)$. □

The subgroup Λ_k of Theorem 1.1 will be

$$\langle\, a_0t, \ldots, a_kt,\ b_1, \ldots, b_8,\ c_1, \ldots, c_8,\ d\,\rangle.$$

2.3. **The equivalence of the presentations.** We will prove:

Proposition 2.2. *P_k and Q_k present the same groups.*

As a first step we establish:

Lemma 2.3. *Mapping $\tau \mapsto t^{-1}$ and $\beta_i \mapsto t^{-1}b_i$ for $1 \le i \le 8$ defines an isomorphism*

$$
\begin{aligned}
&\langle\, \beta_1, \ldots, \beta_8, \tau \ \mid\ {\beta_i}^{-1}\tau\beta_i \ =\ \beta_{i+1}\ (1 \le i \le 7), \quad \beta_8\tau{\beta_8}^{-1} \ =\ \beta_1 \rangle\\
&\qquad \to\ F(b_1, \ldots, b_8) \rtimes_\phi \mathbb{Z} \ =\ \langle\, b_1, \ldots, b_8, t \ \mid\ t^{-1}b_it = \phi(b_i)\ \ (1 \le i \le 8)\,\rangle.
\end{aligned}
$$

Proof. The given map translates the relations ${\beta_i}^{-1}\tau\beta_i = \beta_{i+1}$ $(1 \le i \le 7)$ and $\beta_8\tau{\beta_8}^{-1} \ =\ \beta_1$ to the family

$$
\begin{aligned}
t^{-1}b_it &= \ b_i\, t^{-1}b_{i+1}t \qquad (1 \le i \le 7),\\
t^{-1}b_8t &= \ b_1^{-1}b_8,
\end{aligned}
$$

which is equivalent to $t^{-1}b_it = \phi(b_i)$ $(1 \le i \le 8)$. □

Let P'_k and Q'_k be the presentation obtained from P_k and Q_k by removing all the generators α_i and a_i, respecively, and all the relations in which they occur.

Lemma 2.4. *The groups presented by P'_k and Q'_k are isomorphic via*

$$
\begin{array}{lll}
\tau \mapsto t^{-1}, & \beta_i \mapsto t^{-1}b_i & (1 \le i \le 8),\\
\sigma \mapsto s^{-1}, & \gamma_i \mapsto s^{-1}c_i & (1 \le i \le 8),
\end{array}
$$

where $s = t\,d\,\psi(c_5)$.

Proof. As per Lemma 2.3, translate $\beta_1, \ldots, \beta_8, \tau$ and associated relations to $b_1, \ldots, b_8, t$ and $\gamma_1, \ldots, \gamma_8, \sigma$ and associated relations to $c_1, \ldots, c_8, s$.

The given map converts the relation $\gamma_3\beta_5 = \beta_3\gamma_5$ to
$$s^{-1}c_3t^{-1}b_5 \;=\; t^{-1}b_3s^{-1}c_5.$$
This rearranges as
$$t^{-1}b_5c^{-5}sb_3{}^{-1}t \;=\; c_3{}^{-1}s$$
and then as
$$(t^{-1}b_5t)\,t^{-1}\,s\,(s^{-1}c_5{}^{-1}s)\,t\,(t^{-1}b_3{}^{-1}t) \;=\; s(s^{-1}c_3{}^{-1}s),$$
which is equivalent to
$$\phi(b_5)\,t^{-1}\,s\,\psi(c_5)^{-1}t\,\phi(b_3)^{-1} \;=\; s\,\psi(c_3)^{-1},$$
and so to
$$t^{-1}\,s\,\psi(c_5)^{-1}t \;=\; t^{-1}\,\phi(b_5)^{-1}\,s\,\psi(c_3)^{-1}\,\phi(b_3).$$
So, as $s = t\,d\,\psi(c_5)$,
$$t^{-1}\,d\,t \;=\; t^{-1}\,\phi(b_5)^{-1}\,t\,d\,\psi(c_5)\,\psi(c_3)^{-1}\,\phi(b_3),$$
which gives
$$t^{-1}\,d\,t \;=\; \phi^2(b_5)^{-1}\,d\,\psi(c_5c_3{}^{-1})\,\phi(b_3)$$
as per Q'_k. Next, as $s = t\,d\,\psi(c_5)$, the relation $s^{-1}c_is = \psi(c_i)$ is equivalent to
$$t^{-1}c_it \;=\; d\psi(c_5)\,\psi(c_i)\,\psi(c_5)^{-1}d^{-1}$$
as per Q'_k. □

Inductively define words u_i and v_i for $i \geq 0$ by
$$\begin{array}{rclrcll} u_0 & = & \alpha_k, & u_{i+1} & = & u_i{}^{-1}t^{-1}u_i & (i \geq 0), \\ v_0 & = & a_k, & v_{i+1} & = & v_i{}^{-1}t^{-1}v_it & (i \geq 0). \end{array}$$
The following observation from [6] can be proved by inducting on i.

Lemma 2.5. *On substituting an a_k for each α_k in u_i, the words u_i and $t^{i-1}v_it^{-i}$ become freely equal for all $i \geq 1$.*

Proof of Proposition 2.2. By Lemma 2.4 there is a sequence of Tietze moves carrying the subpresentation P'_k of P_k to Q'_k and the remaining relations (those involving the α_i) to
$$\alpha_1s^{-1}c_1t^{-1} \;=\; t^{-1}s^{-1}c_7\alpha_1, \qquad \alpha_i{}^{-1}t^{-1}\alpha_i \;= \alpha_{i-1},\;\; 1 < i \leq k.$$
A sequence of Tietze moves eliminating $\alpha_1, \ldots, \alpha_{k-1}$ transforms this family to the single relation
$$u_{k-1}s^{-1}c_1t^{-1} \;=\; t^{-1}s^{-1}c_7u_{k-1}.$$
Now substitute an a_k for each α_k. Then, by Lemma 2.5, this relation is equivalent to
$$(t^{k-2}v_{k-1}t^{-(k-1)})s^{-1}c_1t^{-1} \;=\; t^{-1}s^{-1}c_7\,(t^{k-2}v_{k-1}t^{-(k-1)}),$$
which becomes
$$t^{-1}v_{k-1}t \;=\; (t^{-(k-1)}c_7{}^{-1}st^{k-1})\,v_{k-1}\,(t^{-(k-1)}s^{-1}c_1t^{k-1})$$
on conjugating by t^{k-1} and rearranging. A sequence of Tietze moves introducing $a_{k-1}, \ldots, a_1$ expands this to the family
$$t^{-1}a_1t = t^{-(k-1)}c_7{}^{-1}st^{k-1}\,a_1\,t^{-(k-1)}s^{-1}c_1t^{k-1}, \qquad t^{-1}a_it = a_ia_{i-1},\; 1 < i \leq k.$$

The first of these relations becomes $t^{-1}a_1t = a_0$ when we introduce a_0 together with the new relation
$$a_0 \;=\; t^{-(k-1)}c_7{}^{-1}st^{k-1}\,a_1\,t^{-(k-1)}s^{-1}c_1t^{k-1},$$

which becomes $t^{-1}a_0t = ua_1v$ on conjugating by t and eliminating the s and s^{-1} using $s = t\,d\,\psi(c_5)$. $\square$

3. Hyperbolicity

We establish hyperbolicity using techniques employed in [2] and [11].

Consider the presentation 2–complex K_k for P_k assembled from Euclidean unit–squares associated to each of the defining relations with the single exception of $\alpha_1\gamma_1\tau = \tau\gamma_7\alpha_1$ for which we use a Euclidean hexagon made from one unit–square and two equilateral triangles as shown in Figure 1.

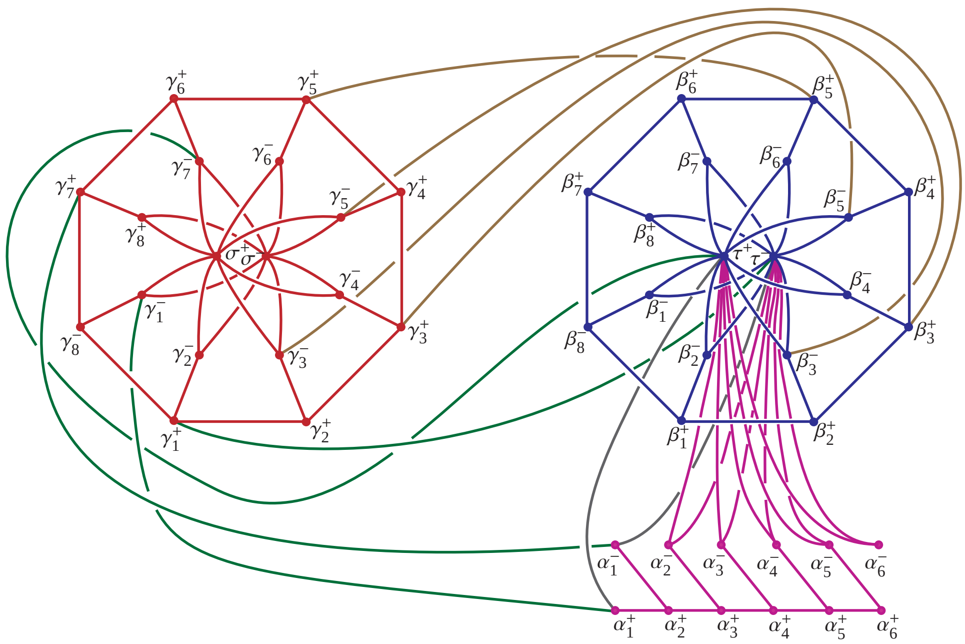

Figure 2. The link of the vertex in the presentation 2–complex associated to the presentation P_6 for Γ_6 given in Section 2. The two grey edges have length $\pi/3$, the four green edges have length $5\pi/6$, and all other edges have length $\pi/2$.

The link in the case $k = 6$ is shown in Figure 2. All edges have length $\pi/2$ apart from the edges τ^+—α_1^+ and τ^-—α_1^- (shown in grey), which have length $\pi/3$, and the edges from γ_1^+—τ^-, γ_7^+—α_1^-, α_1^+—γ_1^-, and τ^+—γ_7^+ (shown in green), which have length $5\pi/6$. Inspecting the link we see that any simple loop in the graph has length at least 2π (separately considering the cases of monochrome and multicoloured simple loops in Figure 1 helps to check this)—that is, for all $k \geq 1$, the link is *large*. So K_k satisfies the *link condition* (see [5]) and its universal cover $\widetilde{K_k}$ is therefore a CAT(0)–space.

To establish that Γ_k is hyperbolic we will show that $\widetilde{K_k}$ contains no subspace isometric to $\mathbb{E}^2$ and then appeal to the Flat Plane Theorem of [4, 8]. The link of a vertex in any isometric copy of $\mathbb{E}^2$ in $\widetilde{K_k}$ would appear as a simple loop of length 2π in the link. But inspecting the link, we find that no edges of length $\pi/3$ or $5\pi/6$ (the grey and green edges) occur in a simple loop of length 2π. Next one can check the edges γ_3^+—β_5^-, γ_3^-—β_3^-, γ_5^-—β_3^+ and γ_5^+—β_5^+ (the brown edges in the figure) do not occur in a simple loop of length 2π. Then it becomes evident that edges occurring in simple loops of length 2π are precisely the edges

$$\begin{array}{llll} \tau^+\text{—}\alpha_i^-, & \tau^-\text{—}\alpha_i^- \ (2 \leq i \leq k), & & \\ \tau^+\text{—}\beta_i^-, & \tau^-\text{—}\beta_i^- \ (1 \leq i \leq 7), & \tau^+\text{—}\beta_8^+, & \tau^-\text{—}\beta_8^+, \\ \sigma^+\text{—}\gamma_i^-, & \sigma^-\text{—}\gamma_i^- \ (1 \leq i \leq 7), & \sigma^+\text{—}\gamma_8^+, & \sigma^-\text{—}\gamma_8^+. \end{array}$$

So every corner of every 2–cell in an isometrically embedded copy of $\mathbb{E}^2$ must give rise to one of the edges in this list. But, looking at the defining relations, we see that no 2–cell in $\widetilde{K_k}$ has this property. Therefore there are no such $\mathbb{E}^2$, and so Γ_k is hyperbolic.

4. Freeness and distortion

4.1. **A family of free–by–cyclic groups.** Fix an integer $l \geq 1$, words u and v on $b_1, \ldots, b_l$, and an automorphism ϕ of $F(b_1, \ldots, b_l)$. Then, for $k \geq 1$, define

$$\Psi_k \ := \ F(a_0, \ldots, a_k, b_1, \ldots, b_l) \rtimes_\theta \mathbb{Z}$$

where θ is the automorphism of $F(a_0, \ldots, a_k, b_1, \ldots, b_l)$ whose restriction to $F(b_1, \ldots, b_l)$ is ϕ and

$$\theta(a_i) \ = \ \begin{cases} ua_1v & i = 0, \\ a_0 & i = 1, \\ a_i a_{i-1} & 1 < i \leq k. \end{cases}$$

Let t denote a generator of the $\mathbb{Z}$–factor, so $t^{-1}a_i t = \theta(a_i)$ and $t^{-1}b_j t = \theta(b_j)$ for all i and j.

The presentation Q_k in Section 2.2 shows Γ_k is an example of such a Ψ_k.

Our aim in the remainder of this section is to establish:

Proposition 4.1. *The subgroup*

$$\Lambda_k \ := \ \langle a_0 t, \ldots, a_k t, b_1, \ldots, b_l \rangle$$

of Ψ_k is free of rank $k + l + 1$ and $\mathrm{Dist}_{\Lambda_k}^{\Psi_k} \geq A_k$.

4.2. **Towards a lower bound on distortion.** In what follows, when, for a word $u = u(a_0, \ldots, a_k, b_1, \ldots, b_l)$, we refer to $\theta^r(u)$, we mean the freely reduced word that equals $\theta^r(u)$ in $F(a_0, \ldots, a_k, b_1, \ldots, b_l)$.

The extreme distortion in the hydra groups of [6] stemmed from the battle between Hercules and the hydra that we described in Section 1. When studying Ψ_k we will need the following more elaborate version of that battle. A *hydra* is now a word on

$$a_0, a_1, \ldots, a_k, b_1, \ldots, b_l$$

in which the a_i only appear with positive exponents. As before, Hercules fights a hydra by removing the first letter. But in this version, the hydra only regenerates after an a_i is removed, and that regeneration is: each remaining a_i and ${b_j}^{\pm 1}$ becomes $\theta(a_i)$ and $\theta({b_j}^{\pm 1})$, respectively. Again, we consider Hercules victorious if, on sufficient repetition, the hydra is reduced to the empty word.

Reprising the example from Section 1, Hercules defeats ${a_2}^3$ as follows:

$$\begin{aligned} {a_2}^3 \ &\to \ (a_2a_1)^2 \ \to \ a_0a_2a_1a_0 \ \to \ a_2a_1a_0ua_1v \ \to \ a_0ua_1v\theta(u)a_0\theta(v) \\ &\to \ a_0\theta(v)\theta^2(u)ua_1v\theta^2(v) \ \to \ a_0\theta(v)\theta^3(v) \ \to \ \varepsilon. \end{aligned}$$

—here, the steps in which Hercules removes a b_j are not shown; the arrows indicate the progression from when an a_i is about to be removed to when an a_i next appears at the front of the word or the hydra becomes the empty word.

The salient point is that a_0 and the b_j play no essential role in this battle; if we removed all b_j and replaced all a_0 by a_1, we would have a battle of the original form. Thus we have the following lemma. [Recall that $\mathcal{H}(u)$ denotes the duration of the battle (of the original type from Section 1 and [6]) against the hydra u.]

Lemma 4.2. *Hercules wins against all hydra w and, in the battle, the number of times he removes an a_i equals $\mathcal{H}(\overline{w})$ where $\overline{w}$ is the word obtained from w by removing all ${b_j}^{\pm 1}$ and replacing all a_0 by a_1.*

Consideration of the original battle between Hercules and the hydra led to the result that, for all $k, n \geq 1$, there is a positive word $u_{k,n} = u_{k,n}(a_1t, \ldots, a_kt)$ of length $\mathcal{H}_k(n)$ that equals ${a_k}^n t^{\mathcal{H}_k(n)}$ in G_k. (This is Lemma 5.1 in [6].) The reason is that the pairing off of a t with an initial a_i in a positive word on $a_1, \ldots, a_k$ corresponds to a decapitation, and the conjugation by t that moves that t into place from the right–hand end causes *regeneration* for the remainder of the word. For example $\mathcal{H}_2(3) = 7$ and

$$\begin{aligned} {a_2}^3 t^7 &= (a_2t)\, t^{-1}{a_2}^2 t\, t^6 \;=\; (a_2t)\,(a_2a_1)^2\, t^6 \;=\; (a_2t)(a_2t)\, t^{-1}a_1a_2a_1t\, t^5 \\ &= \cdots = (a_2t)(a_2t)(a_1t)(a_2t)(a_1t)(a_1t)(a_1t) \;=\; u_{2,3}. \end{aligned}$$

In the corresponding calculation for Ψ_k, only the a_i get paired with t, and on each of the $\mathcal{H}_k(n)$ times that happens, the subsequent conjugation by t can increase length by a factor C which depends only on ϕ, $\ell(u)$ and $\ell(v)$. So:

Lemma 4.3. *There exists $C > 0$ such that for all $k, n \geq 1$, there is a word $\hat{u}_{k,n} = \hat{u}_{k,n}(a_0t, \ldots, a_kt, b_1, \ldots, b_l)$ that equals ${a_k}^n t^{\mathcal{H}_k(n)}$ in Ψ_k and has the properties that*

$$\mathcal{H}_k(n) \;\leq\; \ell(\hat{u}_{k,n}) \;\leq\; C^{\mathcal{H}_k(n)}n$$

and all the (a_it) it contains have positive exponents.

This and our next two lemmas will be components of a calculation that will yield Proposition 4.6 (the analogue of Proposition 5.2 in [6]), which will be the key to establishing a lower bound on the distortion of Λ_k in Ψ_k.

A simple calculation yields:

Lemma 4.4. *$t^{-m}a_1t^{m+1} = \tau_m$ in Ψ_k for all $m \geq 1$ where*

$$\tau_m \;:=\; \begin{cases} a_0t & \text{for } m = 1 \\ \phi^{m-2}(u)\cdots\phi^2(u)u\;(a_1t)\;\phi(v)\phi^3(v)\cdots\phi^{m-1}(v) & \text{for even } m \geq 2 \\ \phi^{m-2}(u)\cdots\phi^3(u)\phi(u)\;(a_0t)\;\phi^2(v)\phi^4(v)\cdots\phi^{m-1}(v) & \text{for odd } m > 2. \end{cases}$$

This combines with

$$t^{-1}(a_1t^{-1})^n \;=\; (t^{-1}a_1t^2)\,(t^{-3}a_1t^4)\,(t^{-5}a_1t^6)\cdots(t^{1-2n}a_1t^{2n})t^{-2n-1}$$

to give:

Lemma 4.5. *There is a constant $C > 0$, depending only on ϕ, $\ell(u)$ and $\ell(v)$, such that for all $n \geq 0$, there is a word $v_n = v_n(a_0t, b_1, \ldots, b_l)$ such that $t^{-1}(a_1t^{-1})^n = v_nt^{-2n-1}$ in Ψ_k, the number of (a_0t) contained in v_n is n and all have positive exponent, and $n \leq \ell(v_n) \leq C^n$.*

Proposition 4.6. *For all $k \geq 2$ and $n \geq 1$, there is a reduced word of length at least $2\mathcal{H}_k(n) + 3$ on $a_0t, a_1t, \ldots, a_kt, b_1, \ldots, b_l$, that equals ${a_k}^n a_2ta_1{a_2}^{-1}{a_k}^{-n}$ in Ψ_k.*

Proof. After rewriting the relation $t^{-1}a_2t = a_2a_1$ as ${a_2}^{-1}ta_2 = t{a_1}^{-1}$, we see ${a_2}^{-1}t^{\mathcal{H}_k(n)}a_2 = (t{a_1}^{-1})^{\mathcal{H}_k(n)}$. So

$${a_k}^n a_2 = \hat{u}_{k,n}a_2(t{a_1}^{-1})^{-\mathcal{H}_k(n)}$$

for $\hat{u}_{k,n}$ as in Lemma 4.3. This gives the first of the equalities

$$\begin{aligned} {a_k}^n a_2 t a_1 {a_2}^{-1} {a_k}^{-n} &= \hat{u}_{k,n} a_2 (t{a_1}^{-1})^{-\mathcal{H}_k(n)} t a_1 (t{a_1}^{-1})^{\mathcal{H}_k(n)} {a_2}^{-1} {\hat{u}_{k,n}}^{-1} \\ &= \hat{u}_{k,n}(a_2 t)\, t^{-1} (t{a_1}^{-1})^{-\mathcal{H}_k(n)} t a_1 (t{a_1}^{-1})^{\mathcal{H}_k(n)} t\, (a_2 t)^{-1} {\hat{u}_{k,n}}^{-1} \\ &= \hat{u}_{k,n}(a_2 t)\, v_{\mathcal{H}_k(n)} t^{-2\mathcal{H}_k(n)-1} t a_1 t^{2\mathcal{H}_k(n)+1} {v_{\mathcal{H}_k(n)}}^{-1}\, (a_2 t)^{-1} {\hat{u}_{k,n}}^{-1} \\ &= \hat{u}_{k,n}(a_2 t)\, v_{\mathcal{H}_k(n)} \tau_{2\mathcal{H}_k(n)} {v_{\mathcal{H}_k(n)}}^{-1}\, (a_2 t)^{-1} {\hat{u}_{k,n}}^{-1}. \end{aligned}$$

The second is a free equality and the third and fourth are applications of Lemmas 4.5 and 4.4, respectively.

This calculation arrives at a word on $a_0 t, a_1 t, \ldots, a_k t, b_1, \ldots, b_l$, that equals ${a_k}^n a_2 t a_1 {a_2}^{-1} {a_k}^{-n}$ in Ψ_k. This word may not be freely reduced, but if we delete all the ${b_j}^{\pm 1}$ it contains, replace all ${a_0}^{\pm 1}$ by ${a_1}^{\pm 1}$, and then freely reduce (i.e. cancel away all $(a_i t)^{\pm 1}(a_i t)^{\mp 1}$ subwords), we get $u_{k,n}\,(a_2 t)\,(a_1 t)\,(a_2 t)^{-1}\,{u_{k,n}}^{-1}$, which has length $2\mathcal{H}_k(n) + 3$. □

4.3. **Freeness and rank.** The result of this section is:

Proposition 4.7. *The subgroup Λ_k is free of rank $k + l + 1$.*

It will be convenient to prove more. In the special case where w represents the identity, the following proposition tells us that there are no non–trivial relations between $a_0 t, \ldots,$ $a_k t, b_1, \ldots, b_l$ and so establishes Proposition 4.7.

Proposition 4.8. *If $w = w(a_0 t, \ldots, a_k t, b_1, \ldots, b_l)$ represents an element of the subgroup $\langle t \rangle$ in Ψ_k, then w freely equals the empty word.*

We begin with an observation on how the groups Ψ_k nest.

Lemma 4.9. *For $1 \le i \le k$, the canonical homomorphism $\Psi_i \to \Psi_k$ is an inclusion.*

Proof. The free–by–cyclic normal forms—a reduced word on $a_0, \ldots, a_k, b_1, \ldots, b_l$ times a power of t—of an element of Ψ_i and its image in Ψ_k are the same. □

We will prove Proposition 4.8 by induction, but first we give a corollary which will be useful in the induction step. We emphasise that when we say that $v(a_0 t, \ldots, a_k t, b_1, \ldots, b_l)$ is *freely reduced* in the following, we mean that there are no $(a_i t)^{\pm 1}(a_i t)^{\mp 1}$ or ${b_j}^{\pm 1}{b_j}^{\mp 1}$ subwords.

Corollary 4.10. *Suppose $v(a_0 t, \ldots, a_k t, b_1, \ldots, b_l)$ is a freely reduced word equalling $\hat{v} t^s$ in Ψ_k where $s \in \mathbb{Z}$ and $\hat{v} = \hat{v}(a_0, \ldots, a_k, b_1, \ldots, b_l)$ is a word in which all the a_i that occur have positive exponents. Then all the $(a_i t)$ in v have positive exponents.*

Proof. When played out against $\hat{v}(a_0, \ldots, a_k, b_1, \ldots, b_l)$, the hydra battle described prior to Lemma 4.2 gives a word $v' = v'(a_0 t, \ldots, a_k t, b_1, \ldots, b_l)$ and an integer s' such that $v' = \hat{v} t^{s'}$ in Ψ_k. Moreover, the exponents of all the $(a_i t)$ in v' are positive. Now, $v^{-1} v' \in \langle t \rangle$ since $\hat{v} = v t^{-s} = v' t^{-s'}$, and so v and v' are freely equal by Proposition 4.8. Therefore the exponents of all the $(a_i t)$ in v are positive. □

Proof of Proposition 4.8. We induct on k. For the base case of $k = 1$, notice that defining $\overline{a}_0 := a_0 t$ and $\overline{a}_1 := a_1 t$, we can transform the presentation

$$\langle\, a_0, a_1, b_1, \ldots, b_l, t \mid t^{-1} a_0 t = u a_1 v,\ t^{-1} a_1 t = a_0,\ t^{-1} b_j t = \phi(b_j)\ \forall j \,\rangle$$

for Ψ_1 to

$$\langle\, \overline{a}_0, \overline{a}_1, b_1, \ldots, b_l, t \mid t^{-1} \overline{a}_0 t = u \overline{a}_1 \phi(v),\ t^{-1} \overline{a}_1 t = \overline{a}_0,\ t^{-1} b_j t = \phi(b_j)\ \forall j \,\rangle,$$

which is an alternative means of expressing Ψ_1 as a free–by–cyclic group from which the result is evident.

For the induction step, we consider a freely reduced word $w = w(a_0t, \ldots, a_kt, b_1, \ldots, b_l)$ representing an element of $\langle t \rangle$ in Ψ_k where $k \geq 2$. If no $(a_kt)^{\pm 1}$ are present in w we can deduce from the induction hypothesis and Lemma 4.9 that w freely reduces to the empty word. For the remainder of our proof we suppose there are $(a_kt)^{\pm 1}$ present, and we seek a contradiction.

Consider shuffling the $t^{\pm 1}$ to the start of w using the defining relations—replacing each a_i and b_j passed by a $t^{\pm 1}$ with $\theta^{\pm 1}(a_i)$ and $\theta^{\pm 1}(b_j)$, respectively. The result will be a power of t times a word on $a_0, \ldots, a_k, b_1, \ldots, b_l$ which freely reduces to the empty word. Such is θ, no a_k are created or destroyed in this process of shuffling the $t^{\pm 1}$. So there is some expression $w_0(a_kt)^{\pm 1}u(a_kt)^{\mp 1}w_1$ for w such that $u = u(a_0t, \ldots, a_{k-1}t, b_1, \ldots, b_l)$ and the ${a_k}^{\pm 1}$ and ${a_k}^{\mp 1}$ in the $(a_kt)^{\pm 1}$ and $(a_kt)^{\mp 1}$ buttressing u cancel after the shuffling and free reduction.

We will address first the case $w = w_0(a_kt)^{-1}u(a_kt)w_1$. Break down the shuffling process by first shuffling the $t^{\pm 1}$ out of w_0, u and w_1, and then carrying the resulting powers to the front of the word:

$$\begin{aligned} w &= w_0\,(a_kt)^{-1}\,u\,(a_kt)\,w_1 \;\;\to\;\; t^{r_0}\,\hat{w}_0\,(a_kt)^{-1}\,t^{r}\,\hat{u}\,(a_kt)\,t^{r_1}\,\hat{w}_1 \\ &\to\;\; t^{r_0+r+r_1}\,\theta^{r+r_1}\,(\hat{w}_0)\,\theta^{r+r_1+1}({a_k}^{-1})\,\theta^{r_1+1}(\hat{u})\,\theta^{r_1+1}(a_k)\,\hat{w}_1 \end{aligned}$$

where $r_0, r, r_1 \in \mathbb{Z}$ and

$$\begin{aligned} \hat{w}_0 &= \hat{w}_0(a_0, \ldots, a_k, b_1, \ldots, b_l), \\ \hat{u} &= \hat{u}(a_0, \ldots, a_{k-1}, b_1, \ldots, b_l), \\ \hat{w}_1 &= \hat{w}_1(a_0, \ldots, a_k, b_1, \ldots, b_l) \end{aligned}$$

are words such that $t^{r_0}\hat{w}_0 = w_0$, $t^r\hat{u} = u$ and $t^{r_1}\hat{w}_1 = w_1$ in Ψ_k. When we expand $\theta^{r+r_1}({a_k}^{-1})$ and $\theta^{r_1+1}(a_k)$ as words on $a_0, \ldots, a_k$, the former ends with an ${a_k}^{-1}$ which must cancel with the a_k at the start of the latter. So $\theta^{r_1+1}(\hat{u})$, and therefore $\hat{u}$, freely equal the empty word. So u represents an element of $\langle t \rangle$ and, by induction hypothesis, freely reduces to the empty word, contrary to the initial assumption that $w(a_0t, \ldots a_kt, b_1, \ldots, b_l)$ is reduced.

In the case $w = w_0(a_kt)u(a_kt)^{-1}w_1$, the shuffling process is

$$\begin{aligned} w &= w_0\,(a_kt)\,u\,(a_kt)^{-1}\,w_1 \;\;\to\;\; t^{r_0}\,\hat{w}_0\,(a_kt)\,t^{r}\,\hat{u}\,(a_kt)^{-1}\,t^{r_1}\,\hat{w}_1 \\ &\to\;\; t^{r_0+r+r_1}\,\theta^{r+r_1}\,(\hat{w}_0)\,\theta^{r+r_1}(a_k)\,\theta^{r_1-1}(\hat{u})\,\theta^{r_1}({a_k}^{-1})\,\hat{w}_1 \end{aligned}$$

where $t^{r_0}\hat{w}_0 = w_0$, $t^r\hat{u} = u$ and $t^{r_1}\hat{w}_1 = w_1$ in Ψ_k, as before. The first and last letters of $\theta^{r+r_1}(a_k)\,\theta^{r_1-1}(\hat{u})\,\theta^{r_1}({a_k}^{-1})$ are a_k and ${a_k}^{-1}$ which cancel, so this subword must freely reduce to the empty word. So $\theta^{r}(a_k)\,\theta^{-1}(\hat{u})\,{a_k}^{-1}$ also freely reduces to the empty word—that is, $\theta^{r+1}(a_k)\,\hat{u}$ freely equals a_ka_{k-1}.

If $r = 0$ then this says that $\hat{u}$ freely equals the empty word and, as before, we have a contradiction. Suppose $r > 0$. Then $\hat{u}^{-1} = (a_ka_{k-1})^{-1}\theta^{r+1}(a_k)$ would be a positive word on $a_0, \ldots, a_{k-1}$ were we to remove all the ${b_1}^{\pm 1}, \ldots, {b_l}^{\pm 1}$ it contains. So, as $\hat{u}^{-1}t^{-r} = u^{-1}$, Corollary 4.10 applies and tells us that u^{-1} would be a positive word were we to remove all the ${b_1}^{\pm 1}, \ldots, {b_l}^{\pm 1}$ it contains. But r is the exponent sum of the $(a_0t)^{\pm 1}, \ldots, (a_{k-1}t)^{\pm 1}$ in u, and so we deduce the contradiction $r \leq 0$. Finally we note that the case $r < 0$ also leads to a contradiction because if we replace w by w^{-1} it becomes the case $r > 0$. □

4.4. **Conclusion.** We deduce from Proposition 4.8 that the word posited in Proposition 4.6 is the *unique* reduced word on $a_0t, \ldots, a_kt, b_1, \ldots, b_l$ that equals ${a_k}^n a_2ta_1{a_2}^{-1}{a_k}^{-n}$ in Ψ_k. This establishes that $\mathrm{Dist}^{\Psi_k}_{\Lambda_k} \geq \mathcal{H}_k$ for all $k \geq 2$. So, by Proposition 1.2 in [6], which says that $\mathcal{H}_k \simeq A_k$ for all $k \geq 1$, we have $\mathrm{Dist}_{\Lambda_k}^{\Psi_k} \succeq A_k$ for all $k \geq 2$. Added to Proposition 4.7, this completes the proof of Proposition 4.1.

Proposition 4.1 applies to the subgroup

$$\langle\, a_0 t, \ldots, a_k t,\ b_1, \ldots, b_8,\ c_1, \ldots, c_8,\ d \,\rangle$$

of Γ_k (presented as Q_k of Section 2) and so, as we established Γ_k to be hyperbolic in Section 3, Theorem 1.1 is proved.

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