HOMOLOGICAL AND HOMOTOPICAL DEHN FUNCTIONS ARE DIFFERENT

AARON ABRAMS, NOEL BRADY, PALLAVI DANI, AND ROBERT YOUNG

Abstract. The homological and homotopical Dehn functions are different ways of measuring the difficulty of filling a closed curve inside a group or a space. The homological Dehn function measures fillings of cycles by chains, while the homotopical Dehn function measures fillings of curves by disks. Since the two definitions involve different sorts of boundaries and fillings, there is no a priori relationship between the two functions, but prior to this work there were no known examples of finitely-presented groups for which the two functions differ. This paper gives the first such examples, constructed by amalgamating a free-by-cyclic group with several Bestvina-Brady groups.

1. Introduction

The classical isoperimetric problem is to determine the maximum area that can be enclosed by a closed curve of fixed length in the plane. This problem has been generalized in many different ways. For example, in a metric space $X$, one can study the homotopical filling area of a curve $\gamma$, denoted $\delta_X(\gamma)$ and defined to be the infimal area of a disk whose boundary is $\gamma$. This leads to the idea of the homotopical Dehn function of $X$, defined as the smallest function $\delta_X$ such that any closed curve $\gamma$ of length $\ell$ has filling area at most $\delta_X(\ell)$. A remarkable result of Gromov [11, 7] states that if $X$ is simply connected and if there is a group $G$ that acts on $X$ geometrically (i.e., cocompactly, properly discontinuously, and by isometries), then the growth rate of $\delta_X$ depends only on $G$; indeed, $\delta_X$ is related to the difficulty of determining whether a product of generators of $G$ represents the identity. We can thus define the Dehn function of a group to be the (homotopical) Dehn function of any simply-connected space that the group acts on geometrically; this is well-defined up to some constants (for details, see Sec. 2).

Another way to generalize the isoperimetric problem is to consider fillings of 1-cycles by 2-chains instead of fillings of curves by disks. If $\alpha$ is a 1-cycle in $X$, we can define its homological filling area $FA(\alpha)$ to be the infimal mass of a 2-chain in $X$ with integer coefficients whose boundary is $\alpha$. This leads to the homological Dehn function $FA_X$, defined as the smallest function such that any 1-chain $\alpha$ of mass at most $\ell$ has a homological filling of area at most $FA_X(\ell)$. Like its homotopical counterpart, $FA_X$ can be used to construct a group invariant: if $H_1(X) = 0$ and if there is a group $G$ that acts on $X$ geometrically, then the growth rate of $FA_X$...
depends only on \( G \) and we can define \( FA_G = FA_X \). Again this is well-defined up to constants.

The exact relationship between these two filling functions has been an open question for some time. The homological Dehn function deals with a wider class of possible fillings (surfaces of arbitrary genus) and a wider class of possible boundaries (sums of arbitrarily many disjoint closed curves), so it is not \textit{a priori} clear whether \( FA_H \) is always the same as \( \delta_H \) when they are both defined. Some hints that they may differ come from a construction of groups with unusual finiteness properties due to Bestvina and Brady [3]. They used a combinatorial version of Morse theory to construct a group which is \( FP_2 \) but not finitely presented. Such a group does not act geometrically on any simply connected space but does acts geometrically on a space with trivial first homology, so its homological Dehn function is defined, but its homotopical Dehn function is undefined.

In this paper, we will construct a family of finitely presented groups such that \( FA_H \) grows strictly slower than \( \delta_H \). Specifically, we will show:

**Theorem 1.1.** For every \( d \in \mathbb{N} \cup \{\infty\} \), there is a CAT(0) group \( G \) containing a finitely presented subgroup \( H \) such that \( FA_H(\ell) \leq \ell^5 \) and the homotopical Dehn function satisfies

\[
\begin{align*}
\ell^d & \leq \delta_H(\ell) \quad & \text{if } d \in \mathbb{N}, \\
e^\ell & \leq \delta_H(\ell) \quad & \text{if } d = \infty.
\end{align*}
\]

**Remark.** Using methods of [5], one can show that in the \( d \in \mathbb{N} \) case, the group \( H \) constructed in the theorem satisfies \( \delta_H(\ell) \leq \ell^{d+3} \).

Our construction uses methods of Brady, Guralnik, and Lee [5] to create a hybrid of a Bestvina-Brady group with a group having large Dehn function. The resulting group is finitely presented, so both \( \delta_H \) and \( FA_H \) are defined, and we will show that the unusual finiteness properties coming from the Bestvina-Brady construction lead to a large gap between homological and homotopical filling functions.

Similar results are known for higher-dimensional versions of \( \delta \) and \( FA \). One can define \( k \)-dimensional homotopical and homological Dehn functions by considering fillings of \( k \)-spheres or \( k \)-cycles by \((k+1)\)-balls or \((k+1)\)-chains; by historical accident, the corresponding homotopical and homological filling functions have come to be called \( FV^k_X \) and \( FV^{k+1}_X \), respectively. The relationship between \( \delta^k_X \) and \( FV^{k+1}_X \) is better understood when \( k \geq 2 \), because in this case the Hurewicz theorem can be used to replace cycles and chains by spheres and balls.

If \( X \) is \( k \)-connected and \( \beta \) is a \((k+1)\)-chain with \( k \geq 2 \), then the Hurewicz theorem can be used to show that \( \beta \) is the image of the fundamental class of a ball under a map \( b : D^{k+1} \to X \) with \( \text{Vol} b = \text{Mass} \beta \). Thus, if \( a : S^k \to X \) is a map of a sphere and \( \alpha \) is the image of the fundamental class of \( S^k \) under \( a \), then \( \delta^k_X(a) = FV^{k+1}_X(\alpha) \), so \( \delta^k_X \leq FV^{k+1}_X \) for \( k \geq 2 \) (see Appendix 2 of [10] and [13]).

Likewise, if \( X \) is \( k \)-connected and \( \alpha \) is a \( k \)-cycle for \( k \geq 3 \), then the Hurewicz theorem can be used to show that \( \alpha \) is the image of the fundamental class of a sphere under a map \( a : S^k \to X \) such that \( \text{Vol} a = \text{Mass} \alpha \) (see Remark 2.6.(4) of [4]). Consequently, since \( \delta^k_X(a) = FV^{k+1}_X(\alpha) \), we have \( FV^{k+1}_X \sim \delta^k_X \) for \( k \geq 3 \).
Thus, if $k \geq 3$ and if $H$ is a group which acts geometrically on a $k$-connected complex, the above results imply that $\delta^k_H \sim FV^{k+1}_H$. When $k = 2$, Young constructed examples of groups for which $\delta^2_H \gneq FV^3_H$ [14]. The examples in this paper are the first known examples of groups for which $\delta^1_H \neq FV^2_H$.

2. Preliminaries

2.1. Dehn functions. For a full exposition of Dehn functions, consult [7]. We will briefly review the definitions that we will need. Let $X$ be a simply connected riemannian manifold or simplicial complex. If $\alpha : S^1 \to X$ is a Lipschitz map, define the homotopical filling area of $\alpha$ to be

$$\delta_X(\alpha) = \inf_{\beta : D^2 \to X, \beta|_{\partial D^2} = \alpha} \text{Area} \beta,$$

where $\beta$ ranges over Lipschitz maps $D^2 \to X$ which agree with $\alpha$ on $\partial D^2$. Since $X$ is simply-connected and any continuous map can be approximated by a Lipschitz map, such maps exist. We can define an invariant of $X$ by letting

$$\delta_X(\ell) = \sup_{\ell(\alpha) \leq \ell} \delta_X(\alpha).$$

We call this the homotopical Dehn function of $X$.

We define a relation on functions $\mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ by $f \sim g$ if there is a $c > 0$ such that for all $n$,

$$f(n) \leq cg(cn + c) + cn + c.$$ 

If $f \sim g$ and $g \sim f$, we write $f \sim g$. Thus $\sim$ distinguishes all functions $x^\mu$ for $\mu \geq 1$, and all functions of the form $\lambda^x$ are equivalent for $\lambda > 1$. Gromov stated (and Bridson proved in [7]) that if $H$ acts geometrically on $X$ (for instance, if $H = \pi_1(M)$ and $X = \tilde{M}$ for some compact $M$), then $\delta_X(n)$ is determined up to $\sim$-equivalence by $H$. If $H$ is finitely presented, then $H$ acts geometrically on the universal cover $\tilde{X}_H$ of a presentation complex, which is a 2-complex $X_H$ with $\pi_1(X_H) = H$. Thus $\delta_H := \delta_{\tilde{X}_H}$ is well-defined up to $\sim$.

To define the homological invariant $FA_H$, suppose that $X$ is a polyhedral complex with $H_1(X) = 0$. If $\alpha$ is a 1-cycle in $X$, we let

$$FA_X(\alpha) = \inf_{\delta \beta = \alpha} \text{Mass} \beta,$$

where Mass $\beta$ is defined to be $\sum |b_i|$ if $\beta = \sum b_i \Delta_i$ is a sum of faces of $X$ with integer coefficients. We can define an invariant of $X$ by letting

$$FA_X(\ell) = \sup_{\text{Mass } \alpha \leq \ell} FA_X(\alpha).$$

We call this the homological Dehn function of $X$. Like the homotopical Dehn function, if $H$ acts geometrically on $X$, then $FA_X$ is determined up to $\sim$ by $H$, and if $X_H$ is a presentation complex for a finitely presented group $H$, we define $FA_H = FA_{\tilde{X}_H}$. 

2.2. Right-angled Artin groups. If \( \Lambda \) is a simple graph (i.e., without loops or multiple edges), we can define a right-angled Artin group (RAAG) based on \( \Lambda \). If \( V(\Lambda) \) and \( E(\Lambda) \) are the vertex set and edge set of \( \Lambda \), we define

\[
A_\Lambda = \langle V(\Lambda) \mid [i(e), t(e)] = 1 \text{ for all } e \in E(\Lambda) \rangle,
\]

where \( i(e) \) and \( t(e) \) are the endpoints of \( e \). We say that \( \Lambda \) is the defining graph of \( A_\Lambda \). These RAAGs generalize free groups and free abelian groups; if \( \Lambda \) is a complete graph, there is an edge between every pair of vertices, so every pair of generators of \( A_\Lambda \) commutes and \( A_\Lambda \) is free abelian. On the other hand, if \( \Lambda \) has no edges, then \( A_\Lambda \) is a free group.

A full exposition of RAAGs can be found in [9]. One important fact that we will use is that for every \( \Lambda \), there is a one-vertex locally CAT(0) cube complex \( X_\Lambda \) with \( \pi_1(X_\Lambda) = A_\Lambda \); this is called the Salvetti complex. This complex can be built directly from the graph \( \Lambda \); it has one vertex, one edge for every vertex of \( \Lambda \), one square for each edge of \( \Lambda \), and one \( n \)-cube for each \( n \)-vertex clique in \( \Lambda \).

Bestvina and Brady [3] used RAAGs to construct subgroups of nonpositively-curved groups with unusual finiteness properties. They defined a homomorphism \( h_{A_\Lambda} : A_\Lambda \to \mathbb{Z} \) which sends each generator of \( A_\Lambda \) to 1 and, viewing \( A_\Lambda \) as the 0-skeleton of \( X_\Lambda \), extended it to a map \( h_{X_\Lambda} : X_\Lambda \to \mathbb{R} \). This map is linear on each cube of \( X_\Lambda \), so the level set \( L_{A_\Lambda} := h_{X_\Lambda}^{-1}(0) \) can be given the structure of a polyhedral complex. The subgroup \( H_{A_\Lambda} := \ker h_{A_\Lambda} \) acts vertex-transitively on \( L_{A_\Lambda} \), so if \( L_{A_\Lambda} \) is connected, the 1-skeleton of \( L_{A_\Lambda} \) is a Cayley graph for \( H_{A_\Lambda} \). In this case, we can construct a generating set for \( H_{A_\Lambda} \) explicitly: each edge of \( L_{A_\Lambda} \) is a diagonal of a square of \( X_\Lambda \), so \( H_{A_\Lambda} \) has a generating set consisting of elements of the form \( ab^{-1} \), where \( a \) and \( b \) are generators of \( A_\Lambda \).

Recall that a complex is flag if every clique of \( n \) vertices spans an \( (n-1) \)-dimensional simplex. Bestvina and Brady proved that the topology of \( \Lambda \) determines the topology of \( h_{X_\Lambda}^{-1}(0) \):

**Theorem 2.1** ([3]). If \( \Lambda \) is the 1-skeleton of a flag complex \( Y \), and \( h_{A_\Lambda}, h_{X_\Lambda} \) are the maps defined above, then \( H_{A_\Lambda} = \ker h_{A_\Lambda} \) acts on the complex \( L_{A_\Lambda} = h_{X_\Lambda}^{-1}(0) \), which is homotopy equivalent to a wedge product of infinitely many copies of \( Y \), indexed by the vertices in \( X_\Lambda \setminus L_{A_\Lambda} \). In fact, \( h_{X_\Lambda}^{-1}(0) \) is a union of infinitely many scaled copies of \( Y \).

The main tool used to prove this theorem is a combinatorial version of Morse theory. If \( X \) is a complex, \( x \in X \) is a vertex, and \( h : X \to \mathbb{R} \) is a function which is linear on each cell and is not constant on any edge, one may define subcomplexes \( \text{Lk}_+(x) \) and \( \text{Lk}_-(x) \) of the link \( \text{Lk}(x) \) called the ascending and descending links of \( x \). To define these, we identify the vertices of \( \text{Lk}(x) \) with the neighborhoods of \( x \).

The ascending link \( \text{Lk}_+(x) \) is the full subcomplex spanned by vertices \( y \) such that \( h(y) > h(x) \) and likewise the descending link \( \text{Lk}_-(x) \) is the full subcomplex spanned by vertices \( y \) such that \( h(y) < h(x) \). These ascending and descending links play a similar role to the ascending and descending manifolds in classical Morse theory.

If \( X \) has one vertex, then all vertices of \( X \) have the same link, so we will write \( \text{Lk}(X) \), \( \text{Lk}_+(X) \), and \( \text{Lk}_-(X) \). The link \( \text{Lk}(X) \) has two vertices \( s^+ \) for each generator \( s \) of \( A_\Lambda \); the ascending link \( \text{Lk}_+(X_\Lambda) \) is spanned by the \( s^+ \)'s, and the descending link \( \text{Lk}_-(X_\Lambda) \) is spanned by the \( s^- \)'s. If \( Y \) is the flag complex with 1-skeleton \( \Lambda \), then \( \text{Lk}_+(X_\Lambda) \) and \( \text{Lk}_-(X_\Lambda) \) are isomorphic to \( Y \).
2.3. **Labeled oriented graph groups.** We can also construct groups using *labeled oriented graphs* (LOGs). A LOG on a set $S$ is a directed multigraph $\Gamma$ with vertex set $S$ and a labeling of the edges given by $l : E(\Gamma) \to S$; loops and multiple edges are allowed. We say that $\Gamma$ presents the group:

$$B_\Gamma := \langle S \mid i(e)^l(e) = t(e), e \in E(\Gamma) \rangle,$$

where the notation $a^b$ represents the conjugation $b^{-1}ab$ and where $i : E(\Gamma) \to S$ and $t : E(\Gamma) \to S$ are the functions taking an edge to its start and end. Since each relation has length 4, the presentation 2-complex $X_\Gamma$ of $B_\Gamma$ is a 2-dimensional cube complex.

Note that although $i(e) = t(e)$ is possible, we may assume that $i(e) \neq l(e) \neq t(e)$ since otherwise we could contract such an edge without changing the group. This implies that $\text{Lk}(X_\Gamma)$ contains no loops or edges of the form $s^+s^-$. As with RAAGs, we can apply Morse theory to LOG groups. Let $h_{B_\Gamma} : B_\Gamma \to \mathbb{Z}$ be the homomorphism mapping each $s \in S$ to 1. This homomorphism can be extended linearly over each cell of $X_\Gamma$ to get a map $h_{X_\Gamma} : X_\Gamma \to \mathbb{R}$. Consider the level set $L_{B_\Gamma} = h_{X_\Gamma}^{-1}(0) \subset X_\Gamma$. As in the RAAG case, the group $\ker h_{B_\Gamma}$ acts on $L_{B_\Gamma}$ vertex-transitively, so if $L_{B_\Gamma}$ is connected, then its 1-skeleton is a Cayley graph for $\ker h_{B_\Gamma}$. Edges in $L_{B_\Gamma}$ are diagonals of squares in $X_\Gamma$, so each orbit of squares labeled $a^b = c$ contributes a generator that can be written as $cb^{-1}$ or $b^{-1}a$. See Fig. 1.

![Figure 1](image.png)

**Figure 1.** The map $h_{X_\Gamma}$ on a 2-cell in $X_\Gamma$. The group element $x(e)$ is in $\ker h$.

As was the case with RAAGs, the link $\text{Lk}(\bar{X}_\Gamma)$ has two vertices $s^\pm$ for each vertex $s$ of $\Gamma$. The ascending link $\text{Lk}^+(\bar{X}_\Gamma)$ is the full subcomplex of $\text{Lk}(\bar{X}_\Gamma)$ spanned by the $s^+$’s, and the descending link $\text{Lk}^-(\bar{X}_\Gamma)$ is spanned by the $s^-$’s. Brady showed:

**Theorem 2.2.** [6, Prop. 2.2.7] Suppose $B_\Gamma$ is a group presented by a LOG $\Gamma$ such that:

- the ascending and descending links $\text{Lk}^+(\bar{X}_\Gamma)$ and $\text{Lk}^-(\bar{X}_\Gamma)$ are trees, and
- the full link $\text{Lk}(\bar{X}_\Gamma)$ has girth at least 4.

Then (1) $X_\Gamma$ is locally $\text{CAT}(0)$, hence a $K(B_\Gamma, 1)$; (2) the level set $L_{B_\Gamma}$ is a tree; and (3) $B_\Gamma$ is isomorphic to the free-by-cyclic group $F_n \rtimes \mathbb{Z}$, where $F_n \cong \ker h_{B_\Gamma}$.

In [5], Brady, Guralnik, and Lee used these groups to construct Stallings-type examples of groups which are of type $F_2$ but not of type $F_3$ and which have Dehn functions with prescribed polynomial or exponential growth rates.
3. Main Theorem

To understand our construction, first consider the problem of constructing a space where the homological and homotopical filling functions differ. Suppose $W$ is a simply-connected space with large Dehn function and $\alpha$ is a closed curve in $W$. In order to reduce the homological filling area but not the homotopical filling area of $\alpha$, we could attach a 2-complex $Z$ to $\alpha$, in which $\alpha$ is the boundary of a 2-chain, but not the boundary of a disk. If $\pi_1(Z)/\langle \alpha \rangle = 0$, the resulting space is still simply connected. By attaching copies of $Z$ to infinitely many closed curves, we can obtain a complex which has large $\delta$ but small FA.

Our construction will be based on a graph of groups with each vertex labeled by one of two groups, $A$ and $Q$. The first group, $A$, will be a right-angled Artin group with a kernel $H_A$ that is $FP_2$ but not finitely presented. This subgroup acts geometrically on a space which has trivial $H_1$ and non-trivial $\pi_1$, which will provide the Z's in the construction.

We define a Thompson complex to be a connected, finite, 2-dimensional flag complex $Y$ whose fundamental group is a simple group with an element of infinite order. (The name comes from the first known group with these properties, Thompson’s group $T$.)

Let $Y$ be a Thompson complex (for example, a triangulation of a presentation complex for Thompson’s group). Note that since $\pi_1(Y)$ is simple, $H_1(Y) = 0$, and every $g \neq 1 \in \pi_1(Y)$ normally generates all of $\pi_1(Y)$. Let $g \in \pi_1(Y)$ be an element of infinite order. By gluing an annulus to $Y$, we may assume that there is a path of length 4 in the 1-skeleton of $Y$ which represents $g$. We label the vertices of this path $a, u, s, v$, and label the rest of the vertices of $Y$ by $y_1, \ldots, y_d$. Since $Y$ is flag, the subcomplex spanned by $a, u, s, v$ must be a cycle of length 4. We will consider the RAAG $A_\Lambda$ where $\Lambda$ is the 1-skeleton of $Y$.

As we won’t need to refer to $\Lambda$ explicitly, we drop it from the notation and set $A = A_\Lambda$. We denote the associated homomorphism by $h_A : A \to \mathbb{Z}$, its extension to a Morse function by $h^{-1}_A : X_A \to \mathbb{R}$, the level set $h^{-1}_A(0)$ by $L_A$, etc. By results of [3], the group $H_A = \ker h_A$ is $FP_2$ but not finitely presented.

The second group, $Q$, will be a product of a LOG group and a free group. Suppose we are given a LOG $\Gamma'$ that satisfies the hypotheses of Theorem 2.2. We may form a new LOG $\Gamma$ by adding an isolated vertex $s$ to $\Gamma'$, adding a loop connecting a vertex $t$ of $\Gamma'$ to itself, and labeling the new edge by $s$. This corresponds to adding a generator $s$ and a relation $[s, t] = 1$ to $B_\Gamma$. We call a LOG $\Gamma$ obtained this way a special LOG or SLOG, and the corresponding group a SLOG group. Note that $\Gamma$ still satisfies the hypotheses of Theorem 2.2.

As with $\Lambda$, we will often omit $\Gamma$ from the notation when it is easily understood. For instance we will abbreviate $B_\Gamma$ by $B$, $X_\Gamma$ by $X_B$, $h_{\Gamma_B}$ by $h_B$, etc.

If $B$ is a SLOG group, then by Theorem 2.2, it can be written as a free-by-cyclic group $B = F_n \rtimes_{\phi} \mathbb{Z}$. (The notation $F_n$ indicates a rank $n$ free group; if we want to emphasize a particular free basis $\{x_i\}$ we will write $F_n(x_1, \ldots, x_n)$.) We define $\text{Dist}_B$ to be the distortion of $F_n$ inside $B$; precisely,

$$\text{Dist}_B(\ell) = \max_g \{|g|_{F_n} : |g|_B \leq \ell\},$$

where $|g|$ is the word length of $g$ in the subscripted group. In [5] there are constructions of SLOG groups $B = F_n \rtimes_{\phi} \mathbb{Z}$ with $\text{Dist}_B \sim e^{\ell}$, and also with $\text{Dist}_B \sim \ell^d$ for all
sufficiently large integers \( d \). The Bieri-Stallings double of \( B \), denoted \( D = B \ast_F B \), has large Dehn function resulting from this distortion; Bridson and Haefliger showed

**Theorem 3.1** ([8], Thm. III.6.20). *If \( B \) and \( D \) are as above, then*

\[
\text{Dist}_B(\ell) \leq \delta_D(\ell).
\]

The group \( D \) will serve as \( W \) in our construction; since its Dehn function is large, it has many curves which are difficult to fill by disks. By an embedding trick appearing in [2], \( D \) can be viewed as a subgroup of the product \( Q := B \times F_2 \), and in fact we will see that \( D \) is the kernel of a map \( h_Q : Q \to \mathbb{Z} \).

We will construct a finitely presented CAT(0) group \( G \) as a graph product of \( Q \) with several copies of \( A \). The subgroup \( H \) will be the kernel of a map \( h : G \to \mathbb{Z} \), and \( H \) will have the structure of a graph product of copies of \( D \) and \( H_A \). We will show that attaching \( H_A \) to \( D \) does not affect \( \delta \), but that the copies of \( Y \) that lie in \( L_A \) can be used to replace fillings by disks with more efficient fillings by chains.

**Theorem 3.2.** Let \( A \) be a RAAG based on a Thompson complex as described above, and let \( B = F_n \rtimes_\phi \mathbb{Z} \) be a SLOG group. Then there exists a finitely presented CAT(0) group \( G \) containing \( A \) and \( B \) such that the homomorphisms \( h_A : A \to \mathbb{Z} \) and \( h_B : B \to \mathbb{Z} \) extend to \( h : G \to \mathbb{Z} \) and such that \( H = \ker(h) \) is finitely presented and satisfies:

\[
\begin{align*}
FA_H(\ell) &\leq \ell^5 \\
\delta_H(\ell) &\geq \text{Dist}_B(\ell).
\end{align*}
\]

Using the examples of SLOG groups constructed in [5], this implies Theorem 1.1.

4. Constructing \( G \) and \( H \)

In this section we construct the groups \( G \) and \( H \) of Theorem 3.2. The construction is similar to the perturbed RAAGs in [5], but we glue several RAAGs (rather than just one) to a free-by-cyclic group.

Throughout this paper, if \( g \) is a group element, \( \overline{g} \) will represent its inverse.

4.1. **The SLOG piece.** Let \( B = B_\Gamma = F_n \rtimes_\phi \mathbb{Z} \) be as in Theorem 3.2. The first step of the construction is to use \( B \) to construct a group \( D \ast Q \cong B \times F_2 \) with large Dehn function. The group \( Q \) will contain several copies of the group \( F_2 \times F_2 \), and we will attach RAAGs \( A_i \) to \( Q \) along some of these groups. The result of this gluing will be \( G \).

Since \( \Gamma \) is a SLOG, it contains an isolated vertex \( s \) which is the label of a single loop in \( \Gamma \). Call the vertex of that loop \( t \). Call the rest of the vertices \( \{a_1, \ldots, a_{n-1}\} \). We have two presentations for \( B \), namely the SLOG presentation with generating set \( \{s, t, a_i\} \), and a free-by-cyclic presentation. For the latter we may take \( \{x_i, t \mid 1 \leq i \leq n\} \) as a generating set, where \( x_i = a_i t \) for \( 1 \leq i < n \) and \( x_n = s t \). (See Fig. 1.) Thus \( B = F_n(\{x_1, \ldots, x_n\} \rtimes_\phi \mathbb{Z} \).\

Let \( D \) be the double \( D = B \ast_{F_n} B \) where the \( F_n \) is generated by the \( x_i \). Theorem 3.1 implies that \( D \) has Dehn function at least as large as \( \text{Dist}_B \). By [2], \( D \) is isomorphic to the subgroup

\[
D \cong F_n(\{x_1, \ldots, x_n\} \rtimes_{(\phi, \psi)} F_2(tu, tv))
\]
of the group $Q = B \times F(u, v)$. Furthermore, if $h_Q : Q \to \mathbb{Z}$ is the group homomorphism taking the elements $a_i, s, t, u, v$ to $1 \in \mathbb{Z}$, then the kernel of $h_Q$ is precisely $D$.

Since $\Gamma$ is a SLOG, the group $B = B_\Gamma$ contains many copies of $F_2$. Recall that the presentation 2-complex $X_B$ of $B$ is a locally CAT(0) 2-dimensional cube complex and thus a $K(B, 1)$. For any $i$, consider the subgroup of $B$ generated by $a_i$ and $s$. It is easy to check that any two of the vertices $a_i^\pm$ and $s^\pm$ in the link of $X_B$ are separated by distance at least 2 in $\text{Lk}(X_B)$. Consequently, the Cayley graph of the subgroup generated by $a_i$ and $s$ is convexly embedded in $\bar{X}_B$. It is thus a copy of $F_2$.

Define $X_Q = X_B \times R_2$, where $R_2$ is a wedge of two circles, so that $X_Q$ is a $K(Q, 1)$. This is locally CAT(0), and by the argument above, for all $i$, the subgroup generated by $a_i, s, u, v$ is a convexly embedded copy of $F_2 \times F_2$.

4.2. Attaching the RAAG pieces. Let $A = A_A$ be a RAAG constructed from a Thompson complex $Y$ as in Sec. 3. Thus $g \in \pi_1(Y)$ has infinite order and is represented by a path $a_\alpha \varepsilon_\beta$ in the defining graph $A$ for $A$. Let $X_A$ be the Salvetti complex of $A$. Since $a, u, s,$ and $v$ span a square in $Y$, $A$ has a convex subgroup isomorphic to $F_2 \times F_2$, generated by $a, u, s, v$. Let $E := F_2 \times F_2$.

We form the group $G$ by gluing copies of $A$ to $Q$ along copies of $E$. Specifically, consider a graph of groups with vertex groups $Q, A_1, \ldots, A_{n-1}$, where $A_i = A$, and with each vertex $A_i$ connected to $Q$ to by an edge. As noted above, for each $i$, the elements $a_i, s, u, v \in Q$ generate a copy of $E$; denote this copy by $E_i$. We identify $E_i$ with the copy of $E$ in $A_i$ by $a_i \leftrightarrow a, u \leftrightarrow u, s \leftrightarrow s, v \leftrightarrow v$. Let $G$ be the fundamental group of this graph of groups. This is a group generated by

$$\{a_1, \ldots, a_{n-1}, s, t, u, v, y_1^j \} \quad i = 1, \ldots, n-1, \ j = 1, \ldots, d.$$ 

We can define subgroups $Q, E_i, A_i$, where $E_i \cong E$, $A_i \cong A$, and

$$Q = B_\Gamma \times F_2(u, v) = \langle a_1, \ldots, a_{n-1}, s, t, u, v \rangle,$$

$$A_i = \langle a_i, s, u, v, y_1^i, \ldots, y_d^i \rangle,$$

$$E_i = A_i \cap Q = F_2(a_i, s) \times F_2(u, v).$$

The homomorphisms $h_Q : Q \to \mathbb{Z}$ and $h_A : A \to \mathbb{Z}$ agree on the edge groups, so we can extend them to a function $h : G \to \mathbb{Z}$.

Let $H = \ker h$.

5. Finite presentability

In this section, we will construct a space on which $H$ acts and consider its topology. This will let us prove that $H$ is finitely presented and will help us bound the Dehn functions of $H$.

We can realize the above construction of $G$ geometrically to construct a $K(G, 1)$ as follows. Let $X_E := R_2 \times R_2$, where $R_2$ is the wedge of two circles; this is a $K(E, 1)$, and each edge group corresponds to copies of $X_E$ in $X_A$ and $X_Q$. Each of these copies of $X_E$ is convex, so we can glue $n - 1$ copies of $X_A$ to $X_Q$ along the $X_E$‘s to obtain a locally CAT(0) cube complex which we call $X_G$. This is a $K(G, 1)$, and the 1-skeleton of its universal cover $\bar{X}_G$ is a Cayley graph of $G$. In particular

Lemma 5.1. The group $G$ is CAT(0).
Now, $H$ is the kernel of the homomorphism $h : G \to \mathbb{Z}$. As before, the vertices of $X_G$ are in correspondence with the elements of $G$, so by viewing $h$ as a function on the vertices of $X_G$, we may extend $h$ linearly over cubes to obtain a Morse function $h : X_G \to \mathbb{R}$. Let $L_G = h^{-1}(0)$.

Since $h$ cuts cubes of $L_G$ “diagonally”, $L_G$ is a polyhedral 2-complex whose cells are slices of cubes. The subgroup $H$ acts freely on $L_G$, and since the vertices of $L_G$ are in 1-to-1 correspondence with the elements of $H$, the action is cocompact and thus geometric. We will show:

**Lemma 5.2.** $L_G$ is simply-connected and thus $H = \ker(h) \subset G$ is finitely presented.

**Proof.** Since $X_G$ is contractible and $h$ is a Morse function on $X_G$, Theorem 4.1 of [3] implies that it is enough to show that the ascending and descending links of any vertex in $X_G$ are simply connected. Since $X_G$ has only one vertex, it is enough to show this for that vertex.

Since the 1-skeleton of $X_G$ is the Cayley graph of $G$, we can label the vertices of $L_k(X_G)$ by $g^\pm$, where $g$ ranges over the generating set $S$. The link $L_k(X_G)$ of the vertex of $X_G$ is obtained by gluing $L_k(X)$ and the various $L_k(X_i)$. For each $1 \leq i \leq n-1$, the links $L_k(X_i)$ each contain a subcomplex with vertices $a_i^\pm$, $u^\pm$, $s^\pm$, and $v^\pm$, and gluing the links along these subcomplexes gives $L_k(X)$.

Likewise, we can form $L_k^+(X_G)$ by gluing $L_k^+(X_A_i)$, $1 \leq i \leq n-1$, to $L_k^+(X_Q)$ along subcomplexes $S_i$ spanned by $a_i^\pm$, $u^\pm$, $s^\pm$, and $v^\pm$. We claim that $L_k^+(X_G)$ is simply-connected. Since $L_k^+(X_A_i)$ is a tree by hypothesis, $L_k^+(X_Q)$ is the suspension of a tree (with suspension points $u^+$ and $v^+$), and thus it is simply-connected.

Since $A_i$ is a right-angled Artin group with defining complex $Y$, each $L_k^+(X_A_i)$ is isomorphic to $Y$, and each $S_i$ is a square such that the normal closure of the subgroup $\pi_1(S_i)$ in $\pi_1(L_k^+(X_A_i)) = \pi_1(Y)$ is all of $\pi_1(Y)$. By the Seifert–van Kampen theorem, $\pi_1(L_k^+(X_G)) = (\pi_1(L_k^+(X_1)) \ast \cdots \ast \pi_1(L_k^+(X_{n-1}))) / \langle \pi_1(S_1), \ldots, \pi_1(S_{n-1}) \rangle = 0$, so the ascending link is simply-connected. The same argument with +’s changed to −’s shows that the descending link is also simply connected, so $L_G$ is simply connected. Thus $H = \pi_1(L_G/H)$ and $H$ is finitely presented. $
$

For an alternate description of $H$, recall that the group $G$ is the fundamental group of a graph of groups, with one vertex labeled $Q$ connected to $n - 1$ vertices labeled $A_i$ by edges labeled $E_i$. Since $G \subset G$, $G$ induces a graph of groups structure on $H$. Indeed, $G$ acts on a tree $T$ whose vertices correspond to the cosets of $Q$ and $A_i$, whose edges correspond to cosets of $E_i$, and whose quotient $G \setminus T$ is a star with $n - 1$ edges. We can restrict the action of $G$ on $T$ to an action of $H$, and since any coset of $Q$, $A_i$, or $E_i$ has nontrivial intersection with $H$, the orbit of any vertex or edge under $H$ is the same as its orbit under $G$. Therefore $H$ acts on $T$ with vertex stabilizers conjugate to $H_Q := H \cap Q$ and $H_{A_i} := H \cap A_i$, edge stabilizers conjugate to $H_{E_i} := H \cap E_i$, and quotient $H \setminus T = G \setminus T$. This shows that $H$ is the fundamental group of the graph of groups with central vertex labeled $H_Q$, connected to $n - 1$ vertices labeled $H_{A_i}$, by edges labeled $H_{E_i}$.

The level set $L_G := h^{-1}(0) \subset X_G$, however, is not the universal cover of a corresponding graph of spaces. To describe $L_G$, we define $L_Q := L_G \cap X_Q$, $L_{A_i} := L_G \cap X_{A_i}$, $L_{E_i} := L_G \cap X_{E_i}$. These level sets have geometric actions by $H_Q$, $H_{A_i}$, and $H_{E_i}$.
\(H_A,\) and \(H_E,\) respectively. We can write the quotient \(L_G/H\) as \(L_Q/H_Q\) with the \(L_A/H_A,\) attached along copies of \(L_E/H_E,\) but since \(L_A,\) and \(L_E,\) are not simply-connected, \(\pi_1(L_A/H_A) \neq A_i\) and \(\pi_1(L_E/H_E) \neq E_i;\) this is a graph of spaces for a different graph of groups.

The fact that \(L_A,\) and \(L_E,\) are not simply-connected will be important in the rest of the paper, so we will go into some more detail. All of the \(L_A,\)'s and all of the \(L_E,\)'s are isometric, so when \(i\) is unimportant, we will denote them by \(L_A\) and \(L_E,\) respectively. To understand the topology of \(L_A\) and \(L_E,\) consider them as subsets of \(\tilde{X}_A.\) By \([3,\) \(L_A\) is a union of scaled copies of \(Y,\) indexed by vertices in \(\tilde{X}_A \setminus L_A.\) Likewise, \(L_E\) is composed of scaled copies of a square, which we denote \(\diamond,\) indexed by vertices in \(\tilde{X}_E \setminus L_E.\) Translating \(L_E\) by elements of \(H_A\) gives infinitely many disjoint copies of \(L_E\) inside \(L_A.\)

By Theorem 8.6 of \([3,\) \(L_A\) is homotopy equivalent to an infinite wedge sum of copies of \(Y\) and \(L_E\) is homotopy equivalent to an infinite wedge sum of copies of \(\diamond,\) so \(L_A\) and \(L_E\) have infinitely generated \(\pi_1.\) Generators of \(\pi_1(L_A\) can be filled in two ways. First, each generator can be freely homotoped into some copy of \(L_E.\) Each copy of \(L_E\) is contained in some \(L_Q,\) and since \(L_Q\) is simply connected, each generator of \(\pi_1(L_A)\) is filled by a disk in one of the copies of \(L_Q.\) Second, though \(\pi_1(L_A)\) is infinitely generated, \(H_1(L_A)\) is trivial, so any curve in \(L_A\) can be filled by some 2-chain entirely inside \(L_A.\) Our goal in the rest of this paper is to use these two types of fillings to show that the homological and homotopical Dehn functions of \(L_G\) are different.

6. Upper bound on the homological Dehn function

In this section we prove the following.

**Proposition 6.1.** With \(H\) as above, we have \(F_{A_H}(\ell) \leq \ell^3.\)

**Proof.** We show that any 1-cycle in \(L_G\) of mass at most \(\ell\) can be filled by a 2-chain of mass \(\leq \ell^3.\) Since \(\ell^3\) is a super-additive function, it is enough to prove this for loops in \(L^{(1)}_G.\)

\(L_G\) consists of copies of \(L_A\) and \(L_Q\) glued together along copies of \(L_E.\) We first show how to homologically fill loops that lie in a single copy of \(L_A\) or \(L_Q,\) and then use these fillings to fill arbitrary loops. Note that each 1-cell of \(L_Q\) is a diagonal of some square in \(\tilde{X}_G,\) so each 1-cell corresponds to a product \(x\bar{y}\) where \(x\) and \(y\) are (certain) generators of \(G.\)

Consider a loop \(\alpha\) of length \(\ell\) that lies in a copy of \(L_A.\) Recall that \(L_A\) is a level set of the Morse function \(h_A : \tilde{X}_A \to \mathbb{R}.\) Since \(\tilde{X}_A\) is CAT(0), there exists a 2-chain \(\beta\) with boundary equal to \(\alpha\) and mass \(\leq \ell^2.\) Further, \(\beta\) lies in \(h_A^{-1}[\ell c, \ell c]\) for some \(c > 0 ([12,\) cf. Prop 2.2 in [1]). We will use \(\beta\) to produce a filling of \(\alpha\) in \(L_A\) using a pushing map as in [1].

Let \(Z\) be the space obtained by deleting open neighborhoods of the vertices of \(X_A\) outside \(L_A,\) with the induced cell structure. So \(Z = X_A \setminus \cup_{v \notin L_A} B^o_{1/4}(v)\). By Theorem 4.2 in [1, there is a \(H_A\)-equivariant Lipschitz retraction (pushing map) \(Q : Z \to L_A\) such that the Lipschitz constant grows linearly with distance from \(L_A.\) Furthermore, if \(S_v = \partial B^o_{1/4}(v)\) is the boundary of one of the deleted neighborhoods, the image of \(S_v,\) is a copy of \(Y\) with metric scaled by a factor of \(h_A(v)\). In particular, if \(\gamma\) is a 1-cycle in \(S_v,\) of length \(\ell(\gamma),\) then \(Q_\gamma(\gamma)\) is a 1-cycle in a scaled copy \(Y_v\) of \(Y.\) Since \(H_1(Y) = 0\) and \(Y\) is compact, the corresponding 1-cycle in \(Y\) has homological
filling area $\leq \ell(\gamma)$. Therefore the original cycle, $Q_2(\gamma)$, has homological filling area $\leq \ell(\gamma)h_A(v)^2$.

Consider the restriction $\beta'$ of $\beta$ to $Z$; this is a 2-chain in $Z$, and $\partial \beta' = \alpha + \sum_{v \in \text{supp} \beta', \gamma_v} \gamma_v$, where $\gamma_v$ is a 1-cycle in $S_v$. We can construct a filling $\beta''$ of $\alpha$ in $L_A$ by combining the image $Q_2(\beta')$ with fillings of each of the $Q_2(\gamma_v)$'s. Since $\beta'$ lies in $h_A^{-1}(-\varepsilon, \varepsilon)$, the restriction of $Q$ to $\text{supp} \beta''$ has $O(\ell)$ Lipschitz constant, and

$$\text{Mass}(\beta'') \leq \ell^2 \text{Mass}(\beta) + \ell^2 \sum_v \ell(\gamma_v) \leq \ell^4 + \ell^2 \sum_v \ell(\gamma_v).$$

Each 2-cell of $\beta$ contributes at most four 1-cells to the $\gamma_v$'s, so $\sum_v \ell(\gamma_v) \leq \ell^2$, and $\text{Mass}(\beta'') \leq \ell^4$, as desired.

Next we produce a quartic mass filling of any loop that lies entirely in a copy of $L_Q$. Such a loop $\alpha$ is labeled by generators of

$$D \cong F_n(x_1, \ldots, x_n) \rtimes \langle \phi, \psi \rangle F_2(tu, tv)$$

where $x_i = a_i \bar{t}$ for $1 \leq i < n$ and $x_n = \bar{s}t$. In this section we will use the notation $t_1 = tu$ and $t_2 = tv$. Let $w$ denote the word labeling $\alpha$, where

$$w = w_1 \ldots w_{\ell}.$$ 

Here the $w_i$ are generators and $w$ represents the identity.

Let $\vartheta : F(x_1, \ldots, x_n) \rtimes F_2(t_1, t_2) \to F(x_1, \ldots, x_n) \rtimes \langle t_1 \rangle$ send $t_2$ to $t_1$ and each other generator to itself. The word $\vartheta(w)$ lies in a free-by-cyclic group which is CAT(0) and therefore has quadratic Dehn function. Thus to fill our loop with quartic mass, it will be enough to reduce $w$ to $\vartheta(w)$ in such a way that the reduction takes quartic mass.

We will achieve this reduction by first decomposing $w$ into subwords as follows.

Let $p : F(x_1, \ldots, x_n) \rtimes F_2(t_1, t_2) \to F_2(t_1, t_2)$ be the projection map. Define

$$w(t) = w_1 \ldots w_t$$

so that $w(0) = w(1) = 1$. Now decompose $w$ as follows:

$$w = [p(w(0))w_1p(w(1))^{-1}][p(w(1))w_2p(w(2))^{-1}] \cdots [p(w(\ell - 1))w_{\ell}p(w(\ell))^{-1}].$$

We can reduce $w$ to $\vartheta(w)$ by reducing each subword in this decomposition to its image under $\vartheta$. If $w_j = t_1^{j+1}t_2^{-j+1}$, then $p(w(j-1))w_jp(w(j))^{-1}$ is freely equal to the identity, and no reduction is necessary. Otherwise, $p(w(j-1)) = p(w(j)) \in F(t_1, t_2)$. Thus it suffices to reduce words of the form $g x_i \bar{g}$, with $g \in F(t_1, t_2)$, or $\vartheta(g x_i \bar{g})$, or equivalently, to fill loops with labels of the form $v = gx_i \bar{g} \vartheta(g \bar{x_i} \bar{g})$.

Write $g = t_i^{d_1} \ldots t_m^{d_m}$ where $r_i$ alternates between 1 and 2 and $d_i \in \mathbb{Z} \setminus \{0\}$. We proceed by induction on $m$.

If $m = 1$ and $g = t_i^d$, then there is nothing to do. If $g = t_i^d$, then $v = t_i^d x_i t_i^{-d} t_i^d x_i t_i^{-d}$ can be written as the sum of four 1-cycles as shown in Fig. 2. Writing $s_1 = s_i u$, the words labeling the 1-cycles are $t_i^{d_1} t_i^{d_1} t_i^{-d_1} t_i^{d_1}$, $s_2^{d_2}(a_i \bar{v})_1 t_i^{d_1} t_i^{d_1} t_i^{-d_1}$, $s_1^{d_1}(a_i \bar{u}) t_i^{d_1} t_i^{-d_1} t_i^{d_1}$, $s_2^{d_2}(a_i \bar{v})_1 t_i^{d_1} t_i^{d_1} t_i^{-d_1}$, and $s_1^{d_1}(a_i \bar{u}) t_i^{d_1} t_i^{-d_1} t_i^{d_1} t_i^{d_1}$. The first and the last are words representing the identity in the CAT(0) group $\langle u \rangle \times \langle s \rangle \times F(u, v)$, and so can be filled with quadratic mass. The middle two are generators of $\pi_1(L_A)$ and can be filled in $L_A$ with a scaled copy of $Y$ with quadratic mass. These four fillings fit together to give a filling of $v$ with quadratic mass.

If $m > 1$, then $g = g_0 t_i^{d_m}$. Let $r = r_m$ and $d = d_m$ and let $g' = g_0 t_i^{d_m}$, (and note that $\vartheta(g) = \vartheta(g')$). As in the $m = 1$ case, we can reduce $t_i^{d_1} x_i t_i^{-d_1}$ to $t_i^{d_1} x_i t_i^{-d_1}$.
using quadratic area. This immediately lets us reduce
\[ gx_i\bar{g}(g\bar{x}_i\bar{g}) \]
to
\[ g'x_i\bar{g}'^{-1}\partial(g'x_i^{-1}\bar{g}'^{-1}). \]

Since we use \( m \) steps, and \( m \leq \ell(g) \leq \ell \), it takes area \( \leq \ell^3 \) (and linear genus) to reduce \( gx_i\bar{g} \) to \( \partial(g\bar{x}_i\bar{g}) \).

Since each of the \( \ell \) subwords in the decomposition of \( w \) above can be reduced to its image under \( \partial \) using mass \( \leq \ell^3 \), the word \( w \) can be reduced to \( \partial(w) \) with mass \( \leq \ell^4 \).

Finally, we consider curves that travel through multiple copies of \( L_A \) and \( L_Q \).

We will need to make arguments based on the graph product decomposition of \( H \), so it will be helpful to have a slightly different complex \( L \) on which \( H \) acts. We construct \( L \) by “stretching” each copy of \( L_E \) in \( L_G \) into a product \( L_E \times [0,1] \). Let \( Z \) be the complex obtained by gluing \( X_Q \) and \( n-1 \) copies of \( X_A \) to \( n-1 \) copies of \( X_E \times [0,1] \) according to the graph product decomposition of \( G \) and let \( \bar{Z} \) be the universal cover of \( Z \). Then the homotopy equivalence \( p : Z \to X_G \) which collapses each copy of \( X_E \times [0,1] \) to a copy of \( X_E \) lifts to a homotopy equivalence \( \bar{p} : \bar{Z} \to X_G \). Then \( H = \ker(h) \subset G \) acts geometrically on the level set \( L = (h \circ \bar{p})^{-1}(0) \), because \( H \) acts geometrically on \( L_G \). The following lemma describes the structure of \( L \).

**Lemma 6.2.** The level set \( L \) intersects each vertex space of \( \bar{Z} \) of the form \( X_Q \) in a copy of \( L_Q \). Likewise, it intersects each vertex space \( X_A \) in a copy of \( L_A \), and each edge space \( X_E \times [0,1] \) in a copy of \( L_E \times [0,1] \).

Each edge in \( L^{(1)} \) either lies in a copy of \( L_Q \), lies in a copy of \( L_A \) for some \( i \), or crosses from \( L_E \times \{0\} \) to \( L_E \times \{1\} \), so we can classify the edges as \( Q \)-edges, \( A \)-edges, or \( E \)-edges. Consider a path of edges in \( L^{(1)} \) of length \( \ell \) and call it \( \alpha \).

By standard arguments (i.e., the normal form theorem for graphs of groups), \( \alpha \) must have an “innermost piece”, i.e., a subpath which enters a copy of \( L_A \), or \( L_Q \).
through a copy of $L_E$, then leaves through the same $L_E$. We write this as $t\gamma t'$, where $t$ and $t'$ lie in the same copy of $L_E \times [0,1]$ and where $\gamma$ is either a path of $A$-edges or a path of $Q$-edges.

Without loss of generality, suppose that the endpoints of $\gamma$ lie in $L_E \times \{0\}$. Call them $(v, 0)$ and $(w, 0)$, and let $\gamma'$ be a geodesic in $L_E \times \{0\}$ from $v$ to $w$. Then $p$ and $\gamma'$ form a loop $\theta$, and since they lie in the union of a copy of $L_E \times \{0,1\}$ and a copy of $L_Q$ or $L_{1,1}$, there is a 2-chain filling $\theta$ whose mass is $\leq (\ell + \ell(\gamma'))(1 + \ell(\gamma'))$. But $L_E$ is undistorted in $L$, since $L_E$ is undistorted in $X_E$ and $X_E$ is convex in $X_G$, so $\ell(\gamma') \leq \ell$, and the filling area of $\theta$ is $\leq \ell^4$.

Repeating this process for the loop $\alpha - \theta$ inductively, we obtain a filling of $\alpha$. Each time we repeat the process, the number of $E$-edges in $\alpha$ decreases by 2, and each time we repeat the process we use a filling of mass $\leq \ell^4$, so the total filling area of $\alpha$ is $\leq \ell^6$.

\[\Box\]

7. Lower bound on homotopical Dehn function

Recall the situation we are in: the group $G$ is a graph product of groups $Q$ and (copies of) $A$ along edge groups $E$, and $\tilde{X}_G, \tilde{X}_Q, \tilde{X}_A, \tilde{X}_E$ are contractible spaces on which respectively $G, Q, A, E$ act geometrically. Meanwhile $H < G$ is a graph product of $D$ (the double of a SLOG group $B$) and copies of $H_A$ (the Bestvina-Brady group associated to the RAAG $A$). In this section we prove the following.

**Theorem 7.1.** With $G$ (hence $H,D,Q,A,B$) as above, we have

$$\delta_H \geq \text{Dist}_B,$$

where $\text{Dist}_B$ is as in Sec. 3.

In order to prove this, we will need the following refinement of Thm. 3.1:

**Lemma 7.2.** For all $\ell > 0$, there is a curve $\gamma : S^1 \to L_Q$ of length $\sim \ell$ such that if $\beta \in C_2(L_Q)$ is a filling of $\gamma$, then

$$\text{Area supp } \beta \geq \text{Dist}_B(\ell),$$

where $\text{supp } \beta$ is the support of $\beta$.

We can take $\gamma$ to be the curve $w_n^{-1}w'_n$ used in the proof of Thm. III.II.6.20 in [8]; the proof in [8] shows that the image of any disk filling $\gamma$ has to have area $\geq \text{Dist}_B(\ell)$, but the same bound applies to any chain filling $\gamma$ as well.

**Proof of Theorem 7.1.** Let $L$ be the complex constructed in the previous section, on which $H$ acts; this is made up of copies of $L_A$ and $L_Q$, joined along copies of $L_E \times [0,1]$. The translates of the set $L_E \times \{1/2\}$ separate $L$ into infinitely many components, each of which is either a copy of $L_A$ glued to copies of $L_E \times [0,1/2]$ or a copy of $L_Q$ glued to copies of $L_E \times [1/2,1]$. Let $L'_A$ and $L'_Q$ be complexes isometric to each type of piece. We will refer to the union of the copies of $L_E \times \{1/2\}$ that lie in $L'_A$ and $L'_Q$ as $\partial L'_A$ and $\partial L'_Q$.

Fix a “root” copy $L_0$ of $L'_Q$ in $L$. Let $\beta : D^2 \to L_0$ be as in Lem. 7.2 and let $\gamma$ be its boundary. Let $\tau : D^2 \to L$ be a filling of $\gamma$ in $L$ and let $[\gamma] \in C_1(L_0; \mathbb{R})$ be the fundamental class of $\gamma$. We will show that there is a chain $\Delta \in C_2(L_0; \mathbb{R})$ which fills $[\gamma]$ and is supported on $L_0 \cap \tau(D^2)$. Then $\Delta - [\beta]$ is a 2-cycle in $L_0$, but
since \( L_0 \) is 2-dimensional and contractible, this implies that \( \Delta = [\beta] \). Therefore, 
\[
\delta_D(\ell) \sim \text{Area } \beta \leq \text{Area } \tau
\]
as desired.

First, we use the structure of \( L \) to break a disk in \( L \) into punctured disks whose images lie in copies of \( L'_Q \) and \( L'_A \). Homotope \( \tau \) so that it is transverse to each copy of \( L_E \times \{1/2\} \). The preimages of the \( L_E \times \{1/2\} \)'s then divide \( D^2 \) into pieces \( M_1, \ldots, M_n \subset D^2 \), and for each \( i \), \( \tau|_{M_i} \) is a punctured disk in a copy of \( L'_A \) or \( L'_Q \).

Each \( M_i \) has a distinguished boundary component which is homotopic to \( \partial D^2 \) in \( D^2 \setminus M_i \), and we call that component the outer boundary of \( M_i \), denoted \( \partial_o M_i \); we call the other boundary curves inner boundaries. Each boundary curve of \( M_i \) either coincides with \( \partial D^2 \) or lies in the preimage of one of the \( L_E \times \{1/2\} \)'s.

Next, we claim that if \( M \) is a punctured disk in \( L'_A \) with boundary in \( \partial L'_A \), the topology of \( L_A \) places strong restrictions on \( M \). More precisely,

**Lemma 7.3.** Suppose that \( f : M \to L'_A \) is a punctured disk in \( L'_A \) with boundary in \( \partial L'_A \). Suppose that \( M \) has a distinguished boundary component \( \partial_o M \) and other boundary components \( \partial_1 M, \ldots, \partial_m M \). If \( [\partial_i M] \in C_1(\partial L'_A; \mathbb{R}) \) is the fundamental class of \( \partial_i M \), then there are \( a_i \in \mathbb{R} \) such that

\[
f_2[\partial_o M] = \sum_{i=1}^m a_i f_2[\partial_i M].
\]

Furthermore, we may assume that \( a_i \neq 0 \) only if \( f(\partial_i M) \) and \( f(\partial_o M) \) lie in the same copy of \( L_E \times \{1/2\} \).

We will prove this lemma at the end of the section, after we use it to construct \( \Delta \). Let \( M_i \) be one of the pieces of \( D^2 \), and suppose that \( \tau \) takes \( \partial_o M_i \) to a curve in \( L_0 \). (For example, take \( M_i \) such that \( \partial_o M_i = \partial D^2 \).

Then \( \tau(M_i) \) is either a punctured disk in \( L_0 \) or a punctured disk in a copy of \( L'_A \) which neighbors \( L_0 \). Let \( M_i \subset D^2 \) be the disk bounded by \( \partial_o M_i \). We claim that there is a chain \( \Delta_i \) in \( \tau(M_i) \cap L_0 \) such that \( \partial \Delta_i = \tau_i[\partial_o M_i] \).

We proceed by induction on the number \( n \) of pieces of \( D^2 \) contained in \( M_i \). If \( n = 1 \), then \( M_i = M_i \). Then, as before, \( \tau(M_i) \) is either a disk in \( L_0 \) or a disk in a copy of \( L'_A \) which neighbors \( L_0 \). In the first case, we can take \( \Delta_i = \tau_i[M_i] \). In the second case, the lemma implies that \( \tau_i[\partial M_i] = 0 \), so we can take \( \Delta_i = 0 \).

Suppose that the claim is true for \( n - 1 \) and let \( M_i \) be a piece of \( D^2 \) such that \( \tau(\partial_o M_i) \subset L_0 \) and \( M_i \) is comprised of \( n \) pieces of \( D^2 \). If \( \tau(M_i) \) is a punctured disk in \( L_0 \), then \( \tau \) takes each inner boundary of \( M_i \) to a curve in \( L_0 \) and each inner boundary bounds a disk in \( D^2 \) with at most \( n - 1 \) pieces. By induction, each \( \tau_i[\partial_j M_i] \) bounds a chain \( \Delta_{ij} \) in \( \tau(M_i) \cap L_0 \). Consequently, we can get the required filling of \( \tau_i[\partial_o M_i] \) as

\[
\Delta_i = \sum_j \Delta_{ij} + \tau_i[M_i].
\]

On the other hand, if \( \tau(M_i) \) is a punctured disk in a copy of \( L'_A \), then Lemma 7.3 implies that we can write

\[
\tau_i[\partial_o M_i] = \sum_j a_{ij} \tau_i[\partial_j M_i],
\]
where each $\partial_i M_i$ is an inner boundary component of $M_i$ which lies in $\partial L_0$. By induction, each of these can be filled by a chain $\Delta_i$ in $\tau(M_i) \cap L_0$, and if

$$\Delta_i = \sum_j a_{ij} \Delta_{ij},$$

then $\Delta_i$ fills $\tau_i[\partial_i M_i]$. 

Therefore, $\tau(M_i) \cap L_0$ supports a chain filling $\gamma$. 

Proof of Lemma 7.3. Let $Y$ be the complex used in the construction of $A$ and $\Diamond \subset Y$ be as in Sec. 5. Choose a basepoint $* \in Y$ such that $* \in \Diamond$. Then by Theorem 8.6 of [3], $L_A^2$ is homotopy equivalent to an infinite wedge sum of copies of $Y$,

$$Y_\infty = \bigvee_{\alpha \in S_A} Y_\alpha$$

(in fact, $S_A$ corresponds to the set of vertices in $X_A \setminus L_A$). Similarly, $\partial L_A^0$ is made up of disjoint copies of $L_E \times \{1/2\}$; call these $L_{E,0}, L_{E,1}, \ldots$, where $f(\partial_0 M) \subset L_{E,0}$. Each of these is homotopy equivalent to a wedge sum indexed by a subset $S_{E,i} \subset S_A$,

$$\Diamond_\infty,i = \bigvee_{\alpha \in S_{E,i}} \Diamond_\alpha.$$ 

The $S_{E,i}$'s partition $S_A$ into disjoint sets. We can consider each of the $\Diamond_\infty,i$'s as a subset of $Y_\infty$, and we can define a homotopy equivalence $h : L_A^2 \to Y_\infty$ that restricts to a homotopy equivalence $L_{E,i} \to \Diamond_\infty,i$ on each $L_{E,i}$.

Let $* \in Y_\infty$ be the basepoint of the wedge sum. Let

$$f' : (M, \partial M) \to (Y_\infty, \bigcup_i \Diamond_\infty,i)$$

be a map which differs from $h \circ f$ by a small homotopy such that $f'^{-1}(*)$ is a graph in $M$. We can further require that if $\lambda$ is a boundary component of $M$ and $f(\lambda) \subset L_{E,i}$, then $f'(\lambda) \subset \Diamond_\infty,i$. Then $f'^{-1}(*)$ cuts $M$ into punctured disks $P_1, \ldots, P_k$, and for each $i$, we can choose an $\alpha_i \in S_A$ such that $f'(P_i) \subset Y_{\alpha_i}$ and $f'(\partial P_i) \subset \Diamond_{\alpha_i}$.

Consider $M$ as a subset of $\mathbb{R}^2$, embedded so that $\partial_0 M$ is the outer boundary of the subset. For each $i$, let $D_i$ be the disk bounded by $\partial_i M$. This embedding lets us choose an outer boundary component $\partial_0 P_i$ for each $P_i$. Furthermore, each closed curve $\lambda$ in $M$ has an inside, and we can use this to put a partial ordering on the boundary curves of the $P_i$'s. If $\lambda$ and $\lambda'$ are two boundary curves, we write $\lambda \prec \lambda'$ if the inside of $\lambda$ is a subset of the inside of $\lambda'$.

If $\lambda_1, \ldots, \lambda_l$ are the inner boundary components of $P_i$, we claim that $f'_i[\partial_0 P_i] \in C_1(\Diamond_{\alpha_i})$ is a linear combination of the $f'_i[\lambda_j]$'s. All of these chains are in fact 1-cycles in $\Diamond_{\alpha_i}$, and since $Z_1(\Diamond_{\alpha_i}) \cong \mathbb{R}$ it's enough to show that if $f'_i[\partial_0 P_i] \neq 0$, then one of the $f'_i[\lambda_j]$'s is nonzero too. But if $f'_i[\lambda_j] = 0$ for all $j$, then $f'(\lambda_j)$ is a null-homotopic curve for all $j$, and so $f'(\partial_0 P_i)$ is the boundary of a disk in $Y_{\alpha_i}$. Since the inclusion $\Diamond \subset Y$ is $\pi_1$-injective, this means that if $f'_i[\lambda_j] = 0$ for all $j$, then $f'_i[\partial_0 M] = 0$ as well, as desired.

We claim that if $\lambda$ is a boundary curve of some $P_i$, then $f'_i[\lambda] \in C_1(\bigcup_i \Diamond_\infty,i)$ is a linear combination of the $f'_i[\partial_0 M]$'s. We proceed by induction on the number of boundary curves $\lambda'$ with $\lambda' \prec \lambda$. If this number is 0, then $\lambda$ is one of the $\partial_i M$, and there's nothing to prove. Otherwise, the inside of $\lambda$ is a union of $P_i$'s and $D_i$'s, so
is a sum of $[\partial P_i]$'s and $[\partial_i M]$'s. By induction, if $P_i$ is inside $\lambda$ and $\lambda \neq \partial_0 P_i$, then $f'_i[\partial P_i]$ is a linear combination of the $f'_i[\partial_i M]$'s, so $f'_i[\lambda]$ is a linear combination of the $f'_i[\partial_i M]$'s too.

Thus, there are $a_i \in \mathbb{R}$ such that

$$f'_i[\partial_0 M] = \sum a_if'_i[\partial_i M],$$

where the equality is taken in $C_1(\bigcup_{i} \ominus_{\infty,i})$, so

$$f'_i[\partial_0 M] = \sum a_i f'_i[\partial_i M]$$

in $H_1(\partial L'_A)$. Since $\partial L'_A$ is 1-dimensional, the equality in fact holds in $C_1(\partial L'_A)$ as well. Finally, $C_1(\partial L'_A) = \bigoplus_i C_1(L_{E,i})$, and if we project the above equation to the $C_1(L_{E,0})$ factor, we get

$$f'_i[\partial_0 M] = \sum_{\{i|f(\partial_i M) \subseteq L_{E,0}\}} a_i f'_i[\partial_i M],$$

as desired. $\square$

References
