Review for the final exam

Final Exam: May 15, 1:30-3:30pm at PHSC 222

Office hour: Monday and Tuesday (May 11, May 12) 10:00-12:30pm
Chapter 1: 1.1-1.6; Chapter 3: 3.1-3.3, 3.5, 3.8; and Chapter 4: 4.1; Total count 30%
Chapter 7: 7.1-7.4. Chapter 8: 8.1-8.2. 70%.

1 First order ODEs

Definition: We start with some terminologies: Differential equations, solutions, general solutions, particular solutions, etc

Exercise 1.1: Check $y = e^{-2x}$ solves

$$y'' - 4y = 0.$$

How to solve a differential equation (1st order): 7 types:

Type 1:

$$\frac{dy}{dx} = f(x).$$

We learned it in Calculus course.

Type 2:

$$\frac{dy}{dx} = g(x)h(y).$$

Separable equations! Change it to

$$\frac{dy}{h(y)} = g(x)dx.$$
Caution: $h(y) = 0$ might be an orphan solution!

**Exercise 1.2:** Solve
\[ \frac{dy}{dx} + x = xy^2. \]

Type 3:
\[ \frac{dy}{dx} + P(x)y = Q(x). \]
Linear. Find integrating factor.

**Exercise 1.3:** Solve
\[ \frac{dy}{dx} + xy = x. \]

Type 4:
\[ \frac{dy}{dx} = F(ax + by + c). \]
Linear substitution: let $v(x) = ax + by + c$.

**Exercise 1.4:** Solve
\[ \frac{dy}{dx} = (x + y + 2)^2. \]

Type 5:
\[ \frac{dy}{dx} = F\left(\frac{y}{x}\right). \]
Nonlinear substitution: let $v(x) = y/x$.

**Exercise 1.5:** Solve
\[ 2xy \frac{dy}{dx} = 4x^2 + 3y^2. \]

Type 6:
\[ \frac{dy}{dx} + P(x)y = Q(x)y^n. \]
Bernoulli equations. Change it to
\[ \frac{dy}{y^n dx} + P(x)y^{-(n-1)} = Q(x). \]
Then let $v(x) = y^{-(n-1)}$.

**Exercise 1.6:** Solve
\[ y^2 \frac{dy}{dx} + 2xy^3 = 6x. \]
Type 7: 
\[ M(x, y)dx + N(x, y)dy = 0. \]

Exact equations.

**Exercise 1.7**: Solve 
\[ (x^3 + \frac{y}{x})dx + (y^2 + \ln x)dy = 0. \]

## 2 Higher order ODEs

**Definition**: We start with some terminologies: Linear differential equations, superposition, general solutions, homogeneous and nonhomogeneous equations, particular solutions, etc.

**Exercise 2.1**: If \( y_1 \) and \( y_2 \) solve equation 
\[ y'' + e^x y' + y\sin x^2 = 0, \]
check \( 2y_1 + 3y_2 \) is also a solution.

**How to solve a homogeneous equation of constant coefficients**: 
\[ y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = 0. \]

Three types.
Consider the Characteristic equation.

**Type 1**: Distinct roots \( s_1, \ldots, s_n \)

**Type 2**: Repeat roots (roots with multiplicity greater than 1)

**Exercise 2.2**: Find and check the general solution to 
\[ y'' + 4y' + 4 = 0. \]

**Type 3**: complex roots.

**Exercise 2.3**: If the characteristic equation of a ODE is
(a). 
\[ (r - 1)^3(r^2 - 4r + 5) = 0, \]
(b). 
\[ (r - 1)^3(r^2 - 2r + 2)^2 = 0, \]
find the general solutions.

How to solve a nonhomogeneous equation of constant coefficients:

\[ y^{(n)} + a_{n-1}y^{(n-1)} + \ldots + a_1y' + a_0y = f(x). \]

Undetermined coefficient method to find one particular solution.

Exercise 2.4: Find the general solution to

\[ y'' + 4y' + 5y = x + e^{2x} \sin x. \]

Variation of parameter method: the formula:

\[ y(x) = -y_1 \int \frac{y_2 f(x)}{W(y_1, y_2)} \, dx + y_2 \int \frac{y_1 f(x)}{W(y_1, y_2)} \, dx. \]

Eigenvalue problems

Exercise 2.5: Find the first positive eigenvalue to

\[ -y'' = \lambda y, \quad y'(0) = y'(\pi) = 0. \]

### 3 Laplace transform

Laplace transform: Definition and basic formulas for \( e^{at}, \cos kt, \sin kt, t^n, \cosh kt, \) sinh \( kt, f'(t), \) translation and partial fraction.

Exercise 3.1: Find: \( \mathcal{L}^{-1}\{ F(s) \} \) if

(a). \( F(s) = \frac{1}{s(s-3)}; \)
(b). \( F(s) = \frac{1}{s^4(s-3)}; \)
(c). \( F(s) = \frac{s^{-1}}{(s+1)^3} \) (hint: translation!).

Using Laplace transform to solve initial value problem: Differential equation to Algebraic equation, solving Algebraic equation, then find \( \mathcal{L}^{-1}\{ F(s) \} \).

Exercise 3.2: Solve:

(a). \( x'' + 4x = \sin 2t, \quad x(0) = x'(0) = 0. \)
(b). \( x' = 4x + 2y, y' = 3x - y; x(0) = 3, \ y(0) = -2. \)

Advanced tricks for laplace transform: Convolution, derivative and integral of transforms.
Exercise 3.3: Find $\mathcal{L}^{-1}\{F(s)\}$ if
(a). $F(s) = \frac{1}{(s^2+9)^2}$;
(b). $F(s) = \ln \frac{s+2}{s+2}$.

Exercise 3.4: Find a nontrivial solution such that $x(0)=0$ to

$$tx'' + (t-2)x' + x = 0.$$ 

Can you find another solution? (Hard, not for everyone).

4 Power series method

Find the recurrence relation, and estimate the radius of convergence. Need to remember Taylor series for $e^x$, $\cos x$, $\sin x$, $\frac{1}{1-x}$, $\ln(1+x)$.

Exercise 4.1 Show that power series method fails to solve

$$x^2 y' = y - x - 1.$$ 

Exercise 4.2: Find one polynomial solution to

$$(1-x^2)y'' - 2xy' + 6y = 0.$$ 

For a power series solution

$$y = \sum_{n=0}^{\infty} c_n x^n$$

will it converge for $x=1/2$?
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