Review for Final Examination

Extra office hour: Dec. 13, 14 10:00-11:30am. Final exam: Dec. 15, 1:30-3:30am

On Integrals (Chapter 5).

Areas: How to find (or define) area; Partition of interval; Area as the limit of summation.

Exercise 1: How to define the area of the region bounded by $y = \sqrt{1 - x^2}$ and $y = 0$?

Definite integrals: The meaning; the notations; algebraic properties; The link to the area (area and net area).

Exercise 2: Is $\int_0^1 f(x)dx = \int_0^1 f(y)dy$? Is $\int_0^1 f(x)dx = \int_0^1 f(a)da$? Is $\int_0^1 f(x + 1)dx = \int_0^1 f(y)dy$? Why?

Fundamental Theorem of Calculus: The contents; The link to indefinite integrals. The net change theorem.

Exercise 3: (Maybe hard) Find

$$\frac{d}{dx} \int_1^{x^2} x \sin t^2 dt.$$

On application (Chapter 6).

Area and volume 6.1 and 6.2: Again, how to set up mathematical formula for area and volume: find the cross-section height or area!

Exercise 4: Find the volume of a spherical cap with height $h$ and radius $r$.

Volume by Cylindrical Shells: Remember the formula.

Exercise 5: Let $S$ be the region bounded by $y = 2x^2 - x^3$ and x-axis, find the volume of the solid obtained by rotating $S$ around the line of $x = 3$. 

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Average: Average formula.

**Chapter 7**

Basic properties about inverse functions: One-one function, symmetric in graphs, relation between derivatives, etc.

**Exercise 6:** \( f(x) = \sqrt{x^3 + x^2 + \ln x + 2} \), find \((f^{-1})'(2) = \).

Logarithmic and exponential functions: The way we introduce these concepts:
Natural Logarithmic function \( y = \ln x = \int_{1}^{x} \frac{1}{t} dt \), \( \Rightarrow \) its inverse natural exponential function \( y = e^x \), \( \Rightarrow \) general exponential function \( y = a^x \) (what is the meaning of \( 2^{\sqrt{2}} \)), \( \Rightarrow \) general logarithmic function \( y = \log_a x \).

**Exercise 7:**
(a). What is the derivative of \( y = x^{x+1} \)? What is the domain of the function? What is \( \lim_{x \to 0^+} x^x \)?
(b). Do you remember why
\[
\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n-1} ?
\]

\( \sin^{-1} x \) and \( \cos^{-1} x \): Domain and range; derivative.

**Exercise 8:** Find \( \sin(\cos^{-1} \frac{1}{2}) \) and \( \tan(\sin^{-1} \frac{1}{3}) \).

**L’Hospital’s rule:** **ALWAYS CHECK CONDITIONS BEFORE YOU USE THE RULE!**

**Exercise 9:**
(a). Find
\[
\lim_{x \to 0^+} \frac{(\ln x)^3}{x^{10}}.
\]
(b). Find
\[
\lim_{x \to +0} (1 + \frac{1}{x})^{4x}.
\]

**Techniques of integration (Chapter 8).**
Integration by parts: Choose your \( u \) and \( v \). Sometimes, you need to integrate by parts twice.

**Exercise 10:** Find the integrals of the following

(a) \( \int x(\ln x)^2 dx \), (b) \( \int xe^{2x} dx \), (c) \( \int x \sin x dx \), (d) \( \int e^x \sin x dx \).
Trigonometric integrals: Three types of integrals.

**Exercise 11:** Find the integrals of the following

(a) $\int \sin^3 x \, dx$,  
(b) $\int \cos^2 x \, dx$,  
(c) $\int \cos 2x \sin x \, dx$.

Trigonometric substitution: $x = a \sin \theta$, etc. Remember to choose $\theta$ in a convenient range (for example: $\theta \in [-\pi/2, \pi/2]$).

**Exercise 12:** Find the integrals of the following

(a) $\int \frac{x^3}{\sqrt{1 - x^2}} \, dx$.

(b) $\int \frac{x^2}{\sqrt{1 - x^2}} \, dx$.

Partial fractions: First, simplify the integrand (improper fraction $\rightarrow$ proper fraction). Then factorize the denominator.

**Exercise 13:** Find the integrals of the following

(a). $\int \frac{x + 2}{4x^2 + 3x - 1} \, dx$.

(b). $\int \frac{1 - 2e^x}{1 - e^{2x}} \, dx$.

Strategy Sometimes you need to combine some techniques together, like in exercise 4 (b). One more example

**Exercise 14:** Find $\int e^{\sqrt{x+1}} \, dx$.

Improper integrals Meanings, computations, comparison for convergence and divergence checking.

**Exercise 15:** Evaluate, if possible the improper integral

$$\int_1^3 \frac{1}{|x - 2|^p} \, dx,$$

where $p$ is a positive constant.
Review for Midterm 1

On Integrals (Chapter 5).

Areas: How to find (or define) area; Partition of interval; Area as the limit of summation.

Exercise 1: How to define the area of the region bounded by $y = \sqrt{1-x^2}$ and $y = 0$?

Definite integrals: The meaning; the notations; algebraic properties; The link to the area (area and net area).

Exercise 2: Is $\int_0^1 f(x) \, dx = \int_0^1 f(y) \, dy$? Is $\int_0^1 f(x) \, dx = \int_0^1 f(a) \, da$? Is $\int_0^1 f(x) \, dx = \int_0^1 f(y) \, d(y+1)$? Is $\int_0^1 f(x+1) \, dx = \int_0^1 f(y) \, dy$? Why?

Fundamental Theorem of Calculus: The contents; The link to indefinite integrals. The net change theorem.

Exercise 3:
(a). We only have a couple of formulae: for example
\[ \int_0^1 \cos x \, dx = \sin x \bigg|_0^1. \]
Why this is true?
(b). Find
\[ \frac{d}{dt} \int_{t+1}^{t+2} x \sin x \, dx. \]
(c). (Hard!) Find
\[ \frac{d}{dx} \int_1^x x \sin t^2 \, dt. \]
(d). (Harder!) Find
\[ \frac{d}{dx} \int_1^{x^2} x \sin t^2 \, dt. \]
Substitution: The way, and the trick to set up your new variable.

Exercise 4: Is \( \int_1^2 f(x)dx = \int_1^2 f(a)d(a + 1) \)?

Exercise 5: If \( f(x) \) is a periodic function with period \( a \) (that is: \( f(x) = f(x+a) \)), show that
\[
\int_0^a f(x)dx = \int_a^{2a} f(x)dx.
\]

On application (Chapter 6).

Area and volume 6.1, 6.2 and 6.3: Again, how to set up mathematical formula for area and volume: find the cross-section height or area; using cylindrical shells!

Exercise 6: Find the volume of a spherical cap with height \( h \) and radius \( r \).

Exercise 7: Find the volume of a solid obtained by rotating about \( x = 3 \) line the region enclosed by \( y = x \) and \( y = x^2 \).

Comments and question to: mzhu@math.ou.edu
Review for Midterm 2

On Integrals (Chapter 6).

Work and Average: The force may change with respect to the location.

Exercise 1. Find the average of $f(x) = \frac{1}{x}$ in $[1, e]$.

Chapter 7

Basic properties about inverse functions: One-one function, symmetric in graphs, relation between derivatives, etc.

Exercise 2:
(a). If $y = f(x)$ is a increasing function in the interval $[1, 5]$, is its inverse function an increasing function? Give your reason.
(b). $f(x) = \sqrt{x^3 + x^2 + x + 1}$, find $(f^{-1})'(2) = $

Logarithmic and exponential functions: The way we introduce these concepts:
Natural Logarithmic function $y = \ln x = \int_1^x \frac{1}{t} dt$, ⇒ its inverse natural exponential function $y = e^x$, ⇒ general exponential function $y = a^x$ (what is the meaning of $2^{\sqrt{2}}$?), ⇒ general logarithmic function $y = \log_a x$.

Exercise 3:
(a). What is the derivative of $y = x^{x+1}$? What is the domain of the function?
(b). Find
$$\int_e^{e^3} \frac{\ln x}{x} dx = $$
(c). Show that
$$\frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n-1}.$$

$\sin^{-1}x$ and $\cos^{-1}x$ Domain and range; derivative.

Exercise 4:
(a). Find $\sin[\cos^{-1} \frac{1}{2}]$ and $\tan[\sin^{-1} \frac{1}{3}]$. 

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(b). Show that
\[ \frac{d(\arcsin x)}{dx} + \frac{d(\arccos x)}{dx} = 0. \]

(c). Show that
\[ \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}. \]

(d). Can you find:
\[ \tan^{-1} x + \cot^{-1} x =? \]

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Midterm 3: Nov. 30. Extra office hour: Nov. 29, 1:30-3:20pm.

Chapter 7

L'Hospital’s rule: **ALWAYS CHECK CONDITIONS BEFORE YOU USE THE RULE!**

**Exercise 1:**
(a). Find \( \lim_{x \to +\infty} \frac{(\ln x)^3}{x} \).

(b). Find \( \lim_{x \to +0} \frac{(\ln x)^3}{x^{10}} \).

(c). Find \( \lim_{x \to +0} (1 + \frac{1}{x})^{4x} \).

Techniques of integration (Chapter 8).

Integration by parts: Choose your \( u \) and \( v \). Sometimes, you need to integrate by parts twice.

**Exercise 2:** Find the integrals of the following

\[
(a) \int x \ln x \, dx, \quad (b) \int xe^x \, dx, \quad (c) \int x \sin x \, dx, \quad (d) \int e^x \sin x \, dx.
\]

Trigonometric integrals: Three types of integrals.

**Exercise 3:** Find the integrals of the following

\[
(a) \int \sin^3 x \, dx, \quad (b) \int \cos^2 x \, dx, \quad (c) \int \cos 2x \sin x \, dx.
\]
Trigonometric substitution: \( x = a \sin \theta \), etc. Remember to choose \( \theta \) in a convenient range (for example: \( \theta \in [-\pi/2, \pi/2] \).

**Exercise 4:** Find the integrals of the following

(a) \[ \int \frac{x}{\sqrt{1-x^2}} \, dx. \]

(b) \[ \int \frac{x^2}{\sqrt{1-x^2}} \, dx. \]

**Partial fractions:** First, simplify the integrand (improper fraction \( \rightarrow \) proper fraction). Then factorize the denominator.

**Exercise 5:** Find the integrals of the following

(a). \[ \int \frac{x + 2}{4x^2 + 3x - 1} \, dx. \]

(b). \[ \int \frac{1 - 2e^x}{1 - e^{2x}} \, dx. \]

**Strategy** Sometimes you need to combine some techniques together, like in exercise 4 (b). One more example

**Exercise 6:** Find \[ \int e^{\sqrt{x}+1} \, dx. \]

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