Conformal Curvature Flows on $S^1$

and Image Processing

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Outline

- Setup of the question: introducing curvatures and curvature flow equations.
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- Outline the main theorems: the steady states, existences and exponential convergence of various flows.
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- Outline the main theorems: the steady states, existences and exponential convergence of various flows.

- Application in image processing.
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Elliptic: The **conformal Laplacian**

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Elliptic: The conformal Laplacian

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L_g = -\frac{4(n-1)}{n-2} \Delta_g + R_g
\]

is a conformal covariant, namely for any \(g_1 \in [g]\), writing \(g_1 = \varphi^{\frac{4}{n-2}} g\) with \(\varphi > 0\), we have

\[
L_{g_1} u = L_{\varphi^{\frac{4}{n-2}} g} u = \varphi^{-\frac{n+2}{n-2}} L_g(\varphi u).
\]
Introduction

The conformal Laplacian \( L_g = -\frac{4(n-1)}{n-2} \Delta_g + R_g \) is not well defined when \( n < 3 \).
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$n = 2$: On $S^2$, if $g = e^{2u}g_s$, then $R_g = e^{-2u}(-\Delta_{g_s}u + 2)$ and the conformal Laplacian is given by $L_g u = -2\Delta_g u + R_g$. Related to the uniformization Theorem.
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- Curvatures, conformal covariant operators?
- Existence of extremal metrics? Parabolic approach?
- Global existences and convergence of the flows?
The $\alpha$-curvature

Let $(S^1, g_s)$ be the unit circle with standard metric $g_s = d\theta \otimes d\theta$. For any metric $g$ on $S^1$, we write $g := d\sigma \otimes d\sigma = v^{-4} g_s$ for some positive function $v$ and define a general $\alpha$-curvature of $g$ for any positive constant $\alpha$ by

$$R_g^\alpha = v^3 (\alpha v_{\theta\theta} + v).$$
The $\alpha$-curvature

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$\alpha$-conformal Laplacian of $g$ is defined by:

$$L_g^\alpha = \alpha \Delta_g + R_g^\alpha,$$

where $\Delta_g = D_{\sigma\sigma}$. 
The $\alpha$-curvature

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$$L^\alpha_g = \alpha \Delta_g + R^\alpha_g,$$

where $\Delta_g = D_{\sigma\sigma}$. $L^\alpha_g$ is a conformal covariant:

**Proposition:** For $\varphi > 0$, if $g_2 = \varphi^{-4} g_1$ then $R^\alpha_{g_2} = \varphi^3 L^\alpha_{g_1} \varphi$, and

$$L^\alpha_{g_2} (\psi) = \varphi^3 L^\alpha_{g_1} (\psi \varphi), \quad \forall \psi \in C^2(S^1).$$
The $\alpha$-curvature

Existence of an extremal metric in the same class by deformation? Parabolic approach: Introduce $\alpha$-curvature flow as

$$\partial_t g = (\overline{R}_g - R_\alpha) g. \tag{2}$$
The \( \alpha \)-curvature

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\partial_t g = (\overline{R}_g^\alpha - R_g^\alpha) g. \tag{2}
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Along the flow we have

\[
\partial_t \overline{R}_g^\alpha = \frac{1}{4\pi} \int (R_g^\alpha - \overline{R}_g^\alpha)^2 d\sigma.
\]
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Will see shortly

$$\overline{R}_g^\alpha \leq 1.$$  

for some special $\alpha$.  

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Affine curvature=$1$-curvature

Two cases $\alpha = 1$ and $\alpha = 4$ are of special interest.
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**Case of $\alpha = 1$.** The affine curvature of a closed strictly convex curve = its 1-curvature: $\mathbf{x}(\theta) \subset \mathbb{R}^2$ ($\theta \in [0, 2\pi]$) be such a curve, parameterized by the angle $\theta$ between the tangent line and $x$-axis. The affine arc-length function $\sigma(\theta) = \int_0^\theta k^{-2/3} \, d\theta$, where $k = k(\theta)$ is the curvature. Then the affine curvature of $\mathbf{x}(\theta)$ is given by:

$$\kappa = k \left( (k^{1/3})_{\theta \theta} + k^{1/3} \right) = \left( k^{1/3} \right)^3 \left( (k^{1/3})_{\theta \theta} + k^{1/3} \right).$$

It coincides with 1-curvature $R_{g_x}^1$, where $g_x = \left( k^{1/3} \right)^{-4} \, d\theta \otimes d\theta$. 
Affine curvature = 1-curvature

One the other hand, given \((S^1, g)\), write \(g = u^{-4}g_s\). Suppose \(u(\theta)\) satisfies the orthogonal condition:

\[
\int_0^{2\pi} \frac{\cos \theta}{u^3(\theta)} d\theta = \int_0^{2\pi} \frac{\sin \theta}{u^3(\theta)} d\theta = 0. \tag{3}
\]

Define \(x(\theta)\) as

\[
x(\theta) = \left( \int_0^{\theta} \frac{\cos \theta}{u^3(\theta)} d\theta, \int_0^{\theta} \frac{\sin \theta}{u^3(\theta)} d\theta \right).
\]

Then \(x(\theta)\) is a closed strictly convex curve, and its affine curvature is equal to \(R_g^1\).
The $\alpha$-curvature

Therefore we have the following correspondence:

\[
\begin{bmatrix}
\text{Close convex curve } x(\theta) \\
\text{with curvature } k(\theta) \\
\text{Affine curvature } \kappa(\theta)
\end{bmatrix}
\leftrightarrow
\begin{bmatrix}
\text{Metric } g = (k^{\frac{1}{3}})^{-4} g_s \text{ on } S^1 \\
\text{with } v = k^{\frac{1}{3}} \text{ satisfies (3).} \\
1\text{-curvature } R_g^1
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Affine curvature $\kappa(\theta)$

1-curvature $R_g^1$

$\alpha = 4$: The 4-curvature $R_g^4$ can be viewed as the scalar curvature in an analogous one-dimensional Yamabe flow.
The $\alpha$-curvature

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$\alpha = 4$: The 4-curvature $R^4_g$ can be viewed as the scalar curvature in an analogous one-dimensional Yamabe flow.

NOW: flow method to find the extremal metrics for the cases $\alpha = 1$ and $\alpha = 4$. We denote $R^1_g$ by $\kappa$. 
1-curvature flow

An analytic proof to the following general Blaschke-Santaló type inequality, which implies that $\overline{K_g}$ is bounded above, and classifies all the extremal metrics:

**General Blaschke-Santaló inequality**

*For $u(\theta) \in H^1(S^1)$ and $u > 0$, if $u$ satisfies the orthogonal condition (3), then*

\[
\int_0^{2\pi} (u^2 - u^2_\theta) d\theta \int_0^{2\pi} u^{-2}(\theta) d\theta \leq 4\pi^2,
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1-curvature flow

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**General Blaschke-Santaló inequality**

For $u(\theta) \in H^1(S^1)$ and $u > 0$, if $u$ satisfies the **orthogonal condition (3)**, then

$$\int_0^{2\pi} (u^2 - u^2_\theta) d\theta \int_0^{2\pi} u^{-2}(\theta) d\theta \leq 4\pi^2,$$

and the equality holds if and only if

$$u(\theta) = c \sqrt{\lambda^2 \cos^2(\theta - \alpha) + \lambda^{-2} \sin^2(\theta - \alpha)}.$$
Theorem: Suppose \( g_0 = u^{-4}(\theta, 0) g_s \) and \( u(\theta, 0) \) satisfies the orthogonal condition (3). Then there is a unique smooth solution to the flow equation

\[
\partial_t g = (\overline{\kappa} - \kappa) g, \quad g(\theta, 0) = g_0(\theta)
\]

for \( t \in [0, +\infty) \). Moreover, \( g(t) \to g_\infty \text{ exponentially} \) as \( t \to +\infty \), and the 1-curvature of \( g_\infty \) is constant.
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for $t \in [0, +\infty)$. Moreover, $g(t) \rightarrow g_\infty$ exponentially as $t \rightarrow +\infty$, and the $1$-curvature of $g_\infty$ is constant.

Our $1$-curvature flow $\Leftrightarrow$ The normalized affine curve-shortening flow,

$$x_t(\sigma, t) = x_{\sigma\sigma}.$$
Outline of the proof

Along the flow $\partial_t g = (\overline{\kappa}_g - \kappa_g)g$, we have

$$\kappa_t = \frac{1}{4} \Delta \kappa + \kappa(\kappa - \overline{\kappa}), \quad u_t = \frac{1}{4} (\kappa - \overline{\kappa})u,$$

i.e.

$$u_t = \frac{1}{4} u^4 \Delta_{gs} u + \frac{1}{4} u^5 - \frac{1}{4} \overline{\kappa} u.$$
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i.e.

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Therefore

$$\partial_t \int d\sigma = \partial_t \int u^{-2} d\theta = \int \frac{1}{2} (\bar{\kappa} - \kappa) d\sigma = 0,$$

$$\partial_t \int_0^{2\pi} u^{-3} (\theta) \cos \theta d\theta = \frac{3\bar{\kappa}}{4} \int_0^{2\pi} u^{-3} (\theta) \cos \theta d\theta.$$

It preserves the length and the orthogonality!
Suppose \( g(t) \) satisfies the flow equation on \([0, T]\), then we have

\[
\kappa_t + \kappa \kappa = \frac{1}{4} \Delta \kappa + \kappa^2 \geq \frac{1}{4} \Delta \kappa.
\]
Existence of the flow

Suppose $g(t)$ satisfies the flow equation on $[0, T]$, then we have

$$\kappa_t + \kappa \kappa = \frac{1}{4} \Delta \kappa + \kappa^2 \geq \frac{1}{4} \Delta \kappa.$$ 

It follows from the maximum principle that

$$\kappa \geq \min \kappa(\sigma, 0) e^{-\int_0^t \kappa d\tau},$$

then:

$$u(\sigma, t) = u(\sigma, 0) \cdot e^{\frac{1}{4} \int_0^t (\kappa - \kappa) d\tau} \geq \tilde{c}_2(\kappa(\sigma, 0), t_0) > 0.$$
Existence of the flow

Suppose $g(t)$ satisfies the flow equation on $[0, T]$, then we have

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Using the evolution equation of $u$ and the orthogonal condition, we can prove that there exist $C = C(T)$ such that $\frac{1}{C} \leq u \leq C$.

Therefore, from standard parabolic estimates that the solution exists for all $t \in [0, \infty)$. 

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Convergence of the flow

For $p \geq 2$, we define

$$F_p(t) = \int_0^{2\pi} |\kappa - \overline{\kappa}|^p d\sigma = ||\kappa - \overline{\kappa}||_{L^p}^p.$$
Convergence of the flow

For $p \geq 2$, we define

$$F_p(t) = \int_0^{2\pi} |\kappa - \bar{\kappa}|^p d\sigma = \|\kappa - \bar{\kappa}\|_{L^p}^p.$$ 

Direct computation yields

$$\partial_t F_p \leq C_1(p)(F_{p+1} + F_p + F_p^{1+\frac{1}{p}}) - C_2(p) \int_0^{2\pi} (|\kappa - \bar{\kappa}|^{\frac{p}{2}})^2 d\sigma.$$
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Using Sobolev inequality and Young’s inequality we have

$$\partial_t F_p \leq C_2(p)(F_p + F_p^\beta + F_p^{1+\frac{1}{p}}) - C_3(p)F_{\frac{3}{3p}}^{\frac{1}{3}},$$  

where $\beta = \frac{2p-1}{2p-3} > 1$ and $C_2, C_3$ are positive constants.
Convergence of the flow

Since \( \partial_t \bar{\kappa}_g = \frac{1}{4\pi} \int (\kappa_g - \bar{\kappa}_g)^2 d\sigma \) and \( \bar{\kappa} \) is bounded above, we have

\[
\int_0^\infty F_2(t) dt < \infty.
\]
Convergence of the flow

Since $\partial_t \overline{\kappa}_g = \frac{1}{4\pi} \int (\kappa_g - \overline{\kappa}_g)^2 d\sigma$ and $\overline{\kappa}$ is bounded above, we have

$$\int_0^{\infty} F_2(t) dt < \infty.$$  

Using above two inequalities, Hölder’s inequality and induction, we can prove the following Lemma:

**Lemma:** For any $p \geq 2$,

$$F_p(t) \to 0 \text{ as } t \to +\infty \text{ and } \int_0^{\infty} F_p(t) dt < \infty.$$
Convergence of the flow

Since \( \partial_t \kappa_g = \frac{1}{4\pi} \int (\kappa_g - \bar{\kappa}_g)^2 d\sigma \) and \( \bar{\kappa} \) is bounded above, we have

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\]

Then we estimate \( \int_0^{2\pi} (\kappa_\sigma)^2 d\sigma \) and obtain that

\[
||\kappa - \bar{\kappa}||_{L^\infty} \to 0, \quad \text{as} \quad t \to \infty.
\]
Exponential Convergence

To show exponential convergence:

\[ \partial_t F_2 \leq \left( -\frac{1}{2} + o(1) \right) F_2, \]

which implies \( F_2 \leq Ce^{-at} \) for some \( C, a > 0 \).
Exponential Convergence

To show exponential convergence:

\[ \partial_t F_2 \leq (\frac{-1}{2} + o(1)) F_2, \]

which implies \( F_2 \leq C e^{-at} \) for some \( C, a > 0 \). Then from the flow equation

\[ u_t = \frac{1}{4} (\kappa - \bar{\kappa}) u, \]

we can prove that \( u(t) \) converges exponentially to some \( u_\infty \) as \( t \to \infty \):

\[ ||u(t) - u_\infty||_{L^\infty} \leq C e^{-at/2}, \]

and the 1-curvature of \( g_\infty := u_\infty^{-4} g_s \) is constant 1. This completes the proof of our main Theorem.
Theorem \((\alpha = 4)\)

For \(\alpha = 4\), we have the similar theorem:

**Theorem:** For an abstract curve \((S^1, u_0^{-4} g_s)\), then there is a unique smooth solution to the flow equation

\[
\partial_t g = (\bar{R}_g - R_g^4)g, \quad g(\theta, 0) = g_0(\theta)
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for \(t \in [0, +\infty)\). Moreover, \(g(t) \to g_\infty\) exponentially as \(t \to +\infty\), and the 4-curvature of \(g_\infty\) is constant.
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for \(t \in [0, +\infty)\). Moreover, \(g(t) \rightarrow g_\infty\) exponentially as \(t \rightarrow +\infty\), and the 4-curvature of \(g_\infty\) is constant.

Along the flow \(\overline{R}_g^4\) is always increasing,

\[
\partial_t \overline{R}_g^4 = \frac{1}{4\pi} \int (R_g^4 - \overline{R}_g^4)^2 d\sigma.
\]
Upper bound for \((\alpha = 4)\)

The upper unbound of \(R_g^4\) follows from the following inequality:

**Proposition:** For \(u(\theta) \in H^1(S^1)\) and \(u > 0\),

\[
\int_0^{2\pi} \left( \frac{1}{4} u^2 - u^2_{\theta} \right) d\theta \int_0^{2\pi} u^{-2}(\theta) d\theta \leq \pi^2,
\]

and the equality holds if and only if

\[
u(\theta) = c \sqrt{\lambda^2 \cos^2 \frac{\theta - \alpha}{2} + \lambda^{-2} \sin^2 \frac{\theta - \alpha}{2}},
\]

for some \(\lambda, c > 0\) and \(\alpha \in [0, 2\pi)\).
For any given $g$ on $S^1$ we write $g = v^{-4/3} g_s$, where $g_s$ is the standard metric. We define general $\alpha$-$Q$-curvature on $(S^1, g)$ as

$$Q_g^\alpha = v^{\frac{5}{3}} \left( \frac{\alpha^2}{9} v_{\theta\theta\theta\theta} + \frac{10\alpha}{9} v_{\theta\theta} + v \right),$$
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$$Q^\alpha_g = v^{5/3} \left( \frac{\alpha^2}{9} v_{\theta\theta\theta\theta} + \frac{10\alpha}{9} v_{\theta\theta} + v \right),$$

and the corresponding operator as

$$P^\alpha_g(f) = \frac{\alpha^2}{9} \Delta^2_g f + \frac{10\alpha}{9} \nabla_g (R^\alpha_g \nabla_g f) + Q^\alpha_g f,$$

where $R^\alpha_g$ is the $\alpha$-curvature of $(S^1, g)$. 

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where $R^\alpha_g$ is the $\alpha$-curvature of $(S^1, g)$.

$P^\alpha_g$ is a conformal covariant: if $g_2 = \varphi^{-4/3}g_1$, then $Q^\alpha_{g_2} = \varphi^{\frac{5}{3}} P^\alpha_{g_1} \varphi$ and $P^\alpha_{g_2} \psi = \varphi^{\frac{5}{3}} P^\alpha_{g_1} (\psi \varphi)$, $\forall \psi$. 
$Q$-curvature flows

Existence of extremal metrics?

Introduce $\alpha$-$Q$-curvature flow

$$\partial_t g = (Q_g^\alpha - Q_g^\alpha)g.$$
**Q-curvature flows**

Existence of extremal metrics?

Introduce $\alpha$-$Q$-curvature flow

$$\partial_t g = (Q^\alpha_g - \overline{Q}^\alpha_g)g.$$ 

In such a setting, the above flow is again a gradient flow of $\overline{Q}^\alpha_g$:

Along the flow, $\overline{Q}^\alpha_g$ is always decreasing:

$$\partial_t \overline{Q}^\alpha_g = -\frac{3}{4\pi} \int_{S^1} (Q^\alpha_g - \overline{Q}^\alpha_g)^2 d\sigma.$$ 

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Again, we are interested in two cases: $\alpha = 1$ and $\alpha = 4$.

Very recently, we prove the existence and convergence of the $\alpha$-$Q$-curvature flow in these two cases.
1-$Q$-curvature flow

Hard part: the proof to the following inequality, which implies that $\overline{Q^1_g}$ is bounded from below, and classifies all the extremal metrics:

Blaschke-Santaló inequality involving higher order derivative?

For $u(\theta) \in H^2(S^1)$ and $u > 0$, satisfying the orthogonal condition

$$
\int_0^{2\pi} \frac{\cos^3 \theta}{u^{5/3} (\theta)} d\theta = \int_0^{2\pi} \frac{\sin^3 \theta}{u^{5/3} (\theta)} d\theta = 0,
$$

(4)

$$
\int_0^{2\pi} \left( u_{\theta\theta}^2 - 10u^2_{\theta} + 9u^2 \right) d\theta \left( \int_0^{2\pi} u^{-2/3} (\theta) d\theta \right)^3 \geq 144\pi^4,
$$

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\int_0^{2\pi} (u_{\theta\theta}^2 - 10u_{\theta}^2 + 9u^2) d\theta \left( \int_0^{2\pi} u^{-2/3}(\theta) d\theta \right)^3 \geq 144\pi^4,
\]

"=" holds if and only if

\[
u_0(\theta) = c \left( \lambda^2 \cos^2(\theta - \beta) + \lambda^{-2} \sin^2(\theta - \beta) \right)^{3/2}.
\]
**Theorem:** Suppose the initial metric $g_0 = v^{-rac{4}{3}}(\theta, 0)g_s$ on $S^1$ satisfies the orthogonal condition (4). Then there is a unique smooth solution to the flow equation

$$\partial_t g = (\overline{Q^1_g} - Q^1_g)g, \quad g(0) = g_0$$

for $t \in [0, +\infty)$. Moreover, $g(t) \to g_\infty$ exponentially as $t \to +\infty$, and the 1-$Q$-curvature of $g_\infty$ is constant.
Direct computation shows that

\[ Q_g^\alpha = \frac{\alpha}{3} \Delta_g R_g^\alpha + (R_g^\alpha)^2, \]

which implies

\[ \int Q_g^\alpha d\sigma = \int (R_g^\alpha)^2 d\sigma. \]
Direct computation shows that

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which implies

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Therefore \( \int (R_{g}^{\alpha})^{2} d\sigma \) is decreasing along the \( \alpha \)-\( Q \)-curvature flow. Thus the \( \alpha \)-\( Q \)-curvature flow can be viewed as one dimensional \textbf{Calabi flow}. \textbf{Caution:} It is not trivial to see that \( \int (R_{g}^{\alpha})^{2} d\sigma \) is bounded below by a positive constant.
Applications in image processing

Top left: original image (with impulse noise).

Top right: affine image flow (10 steps).

Bottom left: affine image flow (20 steps).

Bottom right: our 4-th order image flow (10 steps).

Equation for affine flow:

$$
\Phi_t = \left( \Phi_x^2 \Phi_y + \Phi_y^2 \Phi_x - 2\Phi_x \Phi_y \Phi_{xy} \right)^{\frac{1}{3}}, \quad \Phi(0) = I(x, y).
$$
How does it work?

Consider contour curve for any \( C \geq 0 \):

\[
\Phi(x, y) = C.
\]

Move such curves simultaneously:

\[
\Phi\left((x(t), y(t), t)\right) = C.
\]

So

\[
\Phi_x \cdot x_t + \Phi_y \cdot y_t + \Phi_t = 0,
\]

i.e.

\[
\Phi_t = -(x_t, y_t) \cdot (\Phi_x, \Phi_y).
\]
Let $F(t) = (x(t), y(t))$ represent the curve.

Curve shortening flow is defined by

$$F_t = kN = -k\left(\frac{\Phi_x}{|\nabla \Phi|}, \frac{\Phi_y}{|\nabla \Phi|}\right).$$

$$\Phi_t = \frac{\Phi_x^2 \Phi_{yy} + \Phi_y^2 \Phi_{xx} - 2\Phi_x \Phi_y \Phi_{xy}}{|\nabla \Phi|^2}, \quad \text{on}[0, +\infty) \times \Omega$$

$$\Phi(x, y, 0) = I(x, y), \quad \text{I(x, y) initial image}$$

various boundary condition.
Affine curve shortening flow

Affine shortening flow is defined by

\[ F_t = k^{1/3} N = -k^{1/3} \left( \frac{\Phi_x}{|\nabla \Phi|}, \frac{\Phi_y}{|\nabla \Phi|} \right). \]

\[ \Phi_t = \left( \Phi_x^2 \Phi_{yy} + \Phi_y^2 \Phi_{xx} - 2\Phi_x \Phi_y \Phi_{xy} \right)^{1/3}, \quad \text{on} [0, +\infty) \]

\[ \Phi(x, y, 0) = I(x, y), \quad I(x, y) \text{ initial image} \]

various boundary condition.
Thank you!
Total curvature

For fixed length \( (\int_0^{2\pi} u^{-2} d\theta = 2\pi) \), B-S inequality \( \implies \)

\[ 2\pi \geq \int_0^{2\pi} (u^2 - u_\theta^2) d\theta \]

\[ = \int_0^{2\pi} u^3 (u + u_{\theta\theta}) u^{-2} d\theta \]

\[ = \int_0^{2\pi} \kappa d\sigma. \]

Thus:

\( \bar{\kappa} \leq 1. \)
Affine isoperimetric inequality

B-S inequality $\implies$ for $h > 0$,

$$4\pi^2 \int_0^{2\pi} h(h + h_{\theta\theta})d\theta \geq \left[ \int_0^{2\pi} (h + h_{\theta\theta})^{\frac{2}{3}} d\theta \right]^3.$$ 

Let $h = \langle X, -N \rangle$ be the supporting function of the closed strictly convex curve $X(\theta)$, then $h + h_{\theta\theta} = 1/k$,

$$\int_0^{2\pi} (h + h_{\theta\theta})^{\frac{2}{3}} d\theta = \int_0^{2\pi} k^{-2/3} d\theta = \sigma.$$ 

Also $\int_0^{2\pi} h(h + h_{\theta\theta})d\theta = 2A$.

So

$$8\pi^2 A \geq \sigma^3.$$