Chapter 14 Quadratic and higher-order degree equations (based on factorization)

Introduction

- Completing square
- Quadratic formula (Prop 14.1)
- High order equations

14.1 Completing the square and quadratic formula

Another way to solve a quadratic equation is simply to take the square root on both sides of equation if the equation is in a "good form".

Example 14.1 Solve the equation

$$(x-2)^2 = 9.$$

Solution: Since $9 = 3^2$, we know that

$$x-2=3 \quad \text{or} \quad x-2=-3.$$

Thus,

$$x=5 \quad \text{or} \quad x=-1.$$

$\square$

On the other hand, if we are given an equation such as $x^2-4x-5=0$, we can not apply the above approach directly (though you can use factorization to solve it). We have to manipulate the equation as follows

$$\begin{aligned} x^2-4x-5=0 &\Longleftrightarrow x^2-4x+4-4-5=0 \\ &\Longleftrightarrow (x-2)^2-9=0 \\ &\Longleftrightarrow (x-2)^2=9. \end{aligned}$$

The above procedure is often referred to as *completing square*: writing two terms involving variable x (one term involving x^2, another involving x) and a constant into a perfect square of a linear function by using a formula in Proposition 12.1.

For a general equation with one as the leading coefficient (the coefficient of x^2 term)

$x^2 + bx + c = 0$, we have

$$
\begin{aligned}
x^2 + bx + c = 0 &\Longleftrightarrow \underbrace{x^2 + bx + constant}_{this\ will\ be\ a\ perfect\ square} - \underbrace{the\ same\ constant}_{need\ to\ have\ the\ same\ equation} + c = 0 \\
&\Longleftrightarrow \underbrace{x^2 + bx + (\frac{b}{2})^2}_{(\frac{b}{2})^2\ is\ the\ constant} - (\frac{b}{2})^2 + c = 0 \\
&\Longleftrightarrow (x + \frac{b}{2})^2 - \frac{b^2}{4} + c = 0 \\
&\Longleftrightarrow (x + \frac{b}{2})^2 = \frac{b^2}{4} - c = \frac{b^2 - 4c}{4} \\
&\Longleftrightarrow (x + \frac{b}{2}) = \pm\frac{\sqrt{b^2 - 4c}}{2} \\
&\Longleftrightarrow x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2},
\end{aligned}
$$

where $\pm$ means $+$ or $-$. In the above derivation, we need to assume that $b^2 - 4c \geq 0$. We define $b^2 - 4c$ as the discriminant of the equation, and use Δ to represent it.

We thus obtain a very useful formula

Proposition 14.1. Quadratic formula

For a given equation $x^2 + bx + c = 0$, if its discriminant $\Delta = b^2 - 4c \geq 0$, then its two solutions are given by

$$x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}.$$

Note *It is not only important but essential for students to grasp the quadratic formula by deriving the formula several times by themselves.*

Exercise 14.1 Solve equations by completing square:

1).

$$x^2 - 6x + 5 = 0;$$

2).

$$x^2 - 6x + 7 = 0.$$

Exercise 14.2 We know that $x^2 \geq 0$, thus the minimal value for function $y = x^2$ is zero. Can you find the minimal value for function $y = x^2 + 3x - 5$? Can you find the maximal value for function $y = -x^2 + 4x - 5$?

14.2 Higher-order degree equations

Generally, there is no easy formula for the solutions to a third degree polynomial equation:

$$x^3 + px^2 + qx + r.$$

However, if all coefficients in above equation are integers and above equation has an integer solution $x = n$, then n must be a positive or negative factor of $|r|$. Here is the reason: $x = n$ is a solution, thus for function $f(x) = x^3 + px^2 + qx + r$, $f(n) = 0$. That is

$$0 = n^3 + pn^2 + qn + r.$$

This yields

$$r = n \cdot (-n^2 - pn - q).$$

So $|r|$ is divisible by $|n|$.

Once we find one solution to a third-degree equation, we can use long division to reduce the third-degree equation into a quadratic equation.

Example 14.2 Solve

$$x^3 - 3x^2 + 4 = 0.$$

Solution: We first observe that -1, the negative of factor 1 of number 4, is a solution. Then using long division, we have

$$\begin{aligned} x^3 - 3x^2 + 4 &= (x+1)(x^2 - 4x + 4) && \text{(using long division)} \\ &= (x+1)(x-2)^2. && \text{(continuing to factorize)} \end{aligned}$$

So the original equation becomes as

$$(x+1)(x-2)^2 = 0.$$

Thus

$$x_1 = -1, \quad x_2 = x_3 = 2 \ \ \text{(repeated root)}.$$

□

✍ **Exercise 14.3** Solve

$$x^3 - 2x^2 - x + 2 = 0.$$

My note **Date:**

Chapter 14 Exercises

1. Basic skills: Solving equations by completing the square.

   (a). $x^2 + 4x - 6 = 0$

   (b). $x^2 - 6x + 3 = 0$

   (c). $x^2 - 5x + 2 = 0$

   (d). $4x^2 - 8x + 1 = 0$

2. Basic skills: Solving equations

   (a). $x^2 - 5x = 6$

   (b). $2x^2 - 5x = -2$

   (c). $2x^2 + 7x - 1 = 0$

   (d). $x^3 - 2x^2 - x + 2 = 0$

3. Challenging math. **Vieta's Theorem**: If x_1 and x_2 are two solutions to $x^2 + bx + c = 0$, then

$$x_1 + x_2 = -b, \quad x_1 \cdot x_2 = c.$$

   (a). If $x_1,\ x_2$ are two solutions to $x^2 + \sqrt{5}x - 3 = 0$, find $x_1 + x_2$.

   (b). If $x_1,\ x_2$ are two solutions to $x^2 + \sqrt{5}x - 3 = 0$, find $\frac{1}{x_1} + \frac{1}{x_2}$.