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Uniqueness theorems through the method of moving spheres

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1 Introduction

For $n \geq 2$, $R > 0$, $\bar{x} \in \mathbb{R}^n$, let

$$\mathbb{R}^n_+ = \{\ (x_1, \cdots, x_{n-1}, t) \mid (x_1, \cdots, x_{n-1}) \in \mathbb{R}^{n-1}, t > 0\ \},$$

$$B_R(\bar{x}) = \{\ x \in \mathbb{R}^n \mid |x - \bar{x}| < R\ \}, \qquad B_R = B_R(0),$$

$$B_R^+(\bar{x}) = \{\ (x_1, \cdots, x_{n-1}, t) \in B_R(\bar{x}) \mid t > 0\ \}, \qquad B_R^+ = B_R^+(0).$$

We always use the notation $x = (x', t) \in \mathbb{R}^n_+$.

For $n \geq 3$, $c \in \mathbb{R}$, we consider

$$\begin{cases} -\Delta u &= n(n-2)u^{\frac{n+2}{n-2}} \quad \text{in } \ \mathbb{R}^n_+, \\ \frac{\partial u}{\partial t} &= cu^{\frac{n}{n-2}} \qquad\qquad \text{on } \ \partial\mathbb{R}^n_+. \end{cases} \tag{1}$$

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It is easy to check that for all $\epsilon > 0$, $x_0' \in \mathbb{R}^{n-1}$, and $t_0 = (n-2)^{-1}\epsilon c$, the following functions are solutions of (1):

$$u(x',t) = \left(\frac{\epsilon}{\epsilon^2 + |(x',t) - (x_0',t_0)|^2}\right)^{\frac{n-2}{2}}. \tag{2}$$

Theorem 1.1: Let $u \in C^2(\mathbb{R}_+^n) \cap C^1(\bar{\mathbb{R}}_+^n)$ $(n \geq 3)$ be any nonnegative solution of (1). Then either $u \equiv 0$ or u takes the form (2) for some $\epsilon > 0$, $x_0' \in \mathbb{R}^{n-1}$, and $t_0 = (n-2)^{-1}\epsilon c$.

Almost the same proof applies to the following equation for $n \geq 3$.

$$\begin{cases} -\Delta u &= 0 \quad \text{in} \quad \mathbb{R}_+^n, \\ \frac{\partial u}{\partial t} &= c u^{\frac{n}{n-2}} \quad \text{on} \quad \partial\mathbb{R}_+^n. \end{cases} \tag{3}$$

When $c < 0$, for any $\epsilon > 0$, $x_0' \in \mathbb{R}^{n-1}$ and $t_0 = -(n-2)^{-1}\epsilon c$, the following functions are clearly solutions of (3):

$$u(x',t) = \left(\frac{\epsilon}{(t+t_0)^2 + |x' - x_0'|^2}\right)^{\frac{n-2}{2}}. \tag{4}$$

Theorem 1.2: Let $u \in C^2(\mathbb{R}_+^n) \cap C^1(\bar{\mathbb{R}}_+^n)$ $(n \geq 3)$ be any nonnegative solution of (3). When $c \geq 0$, $u = at + b$ with $a, b \geq 0, a = cb^{n/(n-2)}$. When $c < 0$, either $u \equiv 0$ or u takes the form (4) for some $\epsilon > 0$, $x_0' \in \mathbb{R}^{n-1}$ and $t_0 = -(n-2)^{-1}\epsilon c$.

We also study a two dimensional problem which is similar to (1):

$$\begin{cases} -\Delta u &= e^u \quad \text{in } \mathbb{R}_+^2, \\ \frac{\partial u}{\partial t} &= c e^{u/2} \quad \text{on } \partial\mathbb{R}_+^2. \end{cases} \tag{5}$$

It is easy to see that for any $x_0' \in \mathbb{R}$, $\lambda > 0$, and $t_0 = c\lambda/\sqrt{2}$,

$$u(x',t) = \log \frac{8\lambda^2}{(\lambda^2 + (x'-x_0')^2 + (t-t_0)^2)^2}, \tag{6}$$

satisfies (5) and

$$\int_{\mathbb{R}_+^2} e^u < \infty, \quad \int_{\partial\mathbb{R}_+^2} e^{u/2} < \infty. \tag{7}$$

Theorem 1.3 Let $u \in C^2(\mathbb{R}_+^2) \cap C^1(\bar{\mathbb{R}}_+^2)$ be any solution of (5) satisfying (7). Then u takes the form (6) for some $\lambda > 0$, $x_0' \in \mathbb{R}$ and $t_0 = c\lambda/\sqrt{2}$.

Theorem 1.1 and Theorem 1.2 under a further hypothesis:

$$u(x) = O(\frac{1}{|x|^{n-2}}), \quad \text{for } |x| \text{ large}, \tag{8}$$

were proved by Escobar in [12] in connection with his studies of conformal metrics with prescribed mean curvature on the boundary. Our method is very different from his. We prove it by using the method of moving spheres, a variant of the method of moving planes. Roughly speaking, we make reflection with respect to spheres instead of planes, and then obtain the symmetry of solutions. The method of moving spheres were used by Chou and Chu [10], Padilla [22] , Chen and Li [7]. The method of moving planes goes back to Alexandroff in his study of embedded constant mean curvature surfaces. It was then used and developed through the work of Serrin [25], Gidas, Ni and Nirenberg [15]. More recently further progress has been made, see for example [1], [2], [3], [5], [6], [19], ... and the references therein. After we essentially completed our work, we learned that Terracini [27] had given an alternative proof of Escobar′s result (i.e. Theorem 1.1 and Theorem 1.2 under a further hypothesis (8)) through the method of moving planes. In [9], Chipot, Shafrir and Fila have also presented a proof of Theorem 1.1 and Theorem 1.2. Their original proof did not address the possible singularity at infinity, i.e. they implicitly assumed a decay hypothesis as in Escobar′s result. The authors of this paper showed them the way to handle the possible singularity at infinity. After this paper was submitted, the authors were informed that Theorem 1.2 in the case $c < 0$, was proved independently by Ou in [28]. The case $c > 0$ is much more delicate in, among other things, handling the possible singularity at infinity.

In $\mathbb{R}^n$, the classification of all nonnegative solutions of $-\Delta u = u^{\frac{n+2}{n-2}}$ was given by Obata [24], Gidas, Ni and Nirenberg [15] under (8). The hypothesis (8) was then removed by Caffarelli, Gidas and Spruck [5]. This latter result has played important roles in obtaining energy independent apriori estimates for solutions of Yamabe type equations and scalar curvature equations, see Schoen [26] and Li [21]. In particular, it is used in the work of Schoen [26] in obtaining the compactness of all solutions to the Yamabe problem when the manifold is not conformally equivalent to the standard sphere. This gives an alternative proof of the Yamabe problem, as well as a counting (with multiplicities) of all solutions. Yamabe type problem with prescribed mean curvature on the boundary has been studied by Escobar ([12]-[14]), and existence results are obtained by minimizing the corresponding functionals. In this case, a natural question to ask is whether or not one can obtain compactness results similar to what was done in Schoen′s work [26] for the Yamabe problem. This is the motivation of our present work. We believe that Theorem 1.1 and Theorem 1.2 will be useful in understanding the compactness problem raised above.

Some subcritical problems with nonlinear boundary conditions have been studied by Hu [18], and nonexistence of positive solutions have been proved. Such nonexistence results in $\mathbb{R}^n$ was obtained by Gidas and Spruck [16].

Theorem 1.3 concerns a similar problem in $\mathbb{R}^2_+$. In $\mathbb{R}^2$, the problem was studied by Chen and Li [6], also Chou and Wan [11].

The paper is organized as following. In Section 2, we prove Theorem 1.1-1.2. In fact we only present the proof for Theorem 1.1 since the proof of Theorem 1.2 is very similar. This section is divided into two subsections, one treating the case $c \geq 0$ and the other $c < 0$. The first case is more subtle than the second case, so we give more details in the first case and set the structure of the proof which will be followed in the second case, also in the proof of Theorem 1.3. In Section 3 we prove Theorem 1.3. In the proof we often apply the moving sphere method to the Kelvin transformations of the solutions. This is because we do not know a priori any asymptotic behavior of solutions at infinity. Once we have suitable asymptotic behavior of solutions at infinity, we can work on the solutions directly.

It is easy to see that our approach can be used to give somewhat different proofs of the classification results in $\mathbb{R}^n$.

2 Proof of Theorem 1.1-1.2.

Due to similarity, we only give the proof of Theorem 1.1. If $u = 0$ somewhere in $\bar{\mathbb{R}}^n_+$, it follows from the strong maximum principle and the Hopf lemma (see [25] and [15]) that $u \equiv 0$. Therefore we assume throughout this section that $u > 0$ in $\bar{\mathbb{R}}^n_+$.

2.1 Case $c \geq 0$.

Let u be a positive function satisfying the hypotheses of Theorem 1.1. We perform the Kelvin transformation on u by setting

$$v(x) = \frac{1}{|x|^{n-2}} u(\frac{x}{|x|^2}), \qquad x \in \mathbb{R}^n_+. \tag{9}$$

It follows from elementary calculations that v satisfies

$$\begin{cases} -\Delta v & = n(n-2) v^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n_+, \\ \frac{\partial v}{\partial t} & = c v^{\frac{n}{n-2}} \quad \text{on } \partial \mathbb{R}^n_+ \backslash \{0\}, \\ v & > 0 \quad \text{in } \overline{\mathbb{R}}^n_+ \setminus \{0\}. \end{cases} \tag{10}$$

For $b \in \partial \mathbb{R}^n_+ = \mathbb{R}^{n-1}$, we define the Kelvin transformation of u centered at b by

$$v_b(x) = \frac{1}{|x|^{n-2}} u_b(\frac{x}{|x|^2}),$$

where $u_b(x) = u(x' + b, t)$.

Proposition 2.1 Let u be a positive function satisfying the hypotheses of Theorem 1.1 for some $c \geq 0$. Then for all $b \in \partial \mathbb{R}^n_+ = \mathbb{R}^{n-1}$, there exists $\lambda_b > 0$, such that

$$u_b(x) = \frac{\lambda_b^{2-n}}{|x|^{n-2}} u_b(\frac{\lambda_b^{-2} x}{|x|^2}), \qquad \forall\, x \in \mathbb{R}^n_+.$$

We establish the above proposition by the moving sphere method. From the properties of the Kelvin transformation, we only need to show Proposition 2.1 for x outside $B^+_{\frac{1}{\lambda_b}}(b)$. First we need a lemma which makes it possible to get started.

Lemma 2.1 Let $v \in C^2(\mathbb{R}^n_+) \cap C^1(\bar{\mathbb{R}}^n_+) \setminus \{0\}$ satisfy (10). Then for all $0 < \epsilon < \min\{1, \min_{\partial B_1^+ \cap \partial B_1} v\}$, we have $v(x) \geq \frac{\epsilon}{2(c+1)}$ for all $x \in \overline{B}_1^+ \backslash \{0\}$.

Proof: For $0 < r < 1$, we introduce an auxilary function

$$\varphi_0(x) = \frac{\epsilon}{2(c+1)} - \frac{r^{n-2}\epsilon}{|x|^{n-2}} + \frac{\epsilon t}{2}, \qquad x \in \overline{B}_1^+ \setminus B_r^+.$$

Considering $w = v - \varphi_0$. Clearly w satisfies

$$\begin{cases} \Delta w & \leq 0 \qquad\qquad \text{in } B_1^+ \backslash \overline{B}_r^+, \\ \frac{\partial w}{\partial t} & = cv^{\frac{n}{n-2}} - \frac{\epsilon}{2} \qquad \text{on } \partial(\overline{B}_1^+ \backslash B_r^+) \cap \partial\mathbb{R}_+^n. \end{cases} \tag{11}$$

We will show that

$$w \geq 0 \quad \text{in } \overline{B}_1^+ \setminus B_r^+. \tag{12}$$

On $\partial B_r^+ \cap \partial B_r$, $w = v - (\frac{\epsilon}{2(c+1)} - \epsilon + \frac{\epsilon t}{2}) > v > 0$. On $\partial B_1^+ \cap \partial B_1$, $w = v - (\frac{\epsilon}{2(c+1)} - \frac{r^{n-2}\epsilon}{|x|^{n-2}} + \frac{\epsilon t}{2}) > v - \epsilon > 0$. Suppose the contrary of (12), it follows from the maximum principle that there exists some $x_0 = (x_0', 0)$ with $r < |x_0'| < 1$ such that $w(x_0) = \min\limits_{\overline{B}_1^+ \backslash B_r^+} w < 0$. It follows that $\frac{\partial w}{\partial t}(x_0) \geq 0$. Using the boundary condition of w, we have $v(x_0) \geq (\frac{\epsilon}{2(c+1)})^{(n-2)/n} > \frac{\epsilon}{2(c+1)}$. It follows that

$$w(x_0) = v(x_0) - (\frac{\epsilon}{2(c+1)} - \frac{r^{n-2}\epsilon}{|x_0'|^{n-2}}) > v(x_0) - \frac{\epsilon}{2(c+1)} > 0,$$

which contradicts to $w(x_0) < 0$. Thus we have (12). For $x \in B_1^+ \setminus \{0\}$, it follows from (12) that for all $0 < r < |x|$ we have $w(x) \geq 0$. Let $r \to 0$, Lemma 2.1 follows.

Corollary 2.1 (scaled version) Let $v \in C^2(\mathbb{R}_+^n) \cap C^1(\bar{\mathbb{R}}_+^n) \setminus \{0\}$ satisfy (10). Then for all $0 < \epsilon < \min\{R^{(2-n)/2}, \min\limits_{\partial B_R^+ \cap \partial B_R} v\}$, we have $v(x) \geq \frac{\epsilon}{2(c+1)}$ for all $x \in \overline{B}_R^+ \backslash \{0\}$.

Proof: We just apply Lemma 2.1 to $\bar{v}(x) = R^{\frac{n-2}{2}} v(Rx)$.

Proof of Proposition 2.1: Define for $\lambda > 0, b \in \partial\mathbb{R}_+^n = \mathbb{R}^{n-1}$,

$$w_{\lambda,b}(x) = v_b(x) - \frac{\lambda^{n-2}}{|x|^{n-2}} v_b(\frac{\lambda^2 x}{|x|^2}).$$

In the following we always write $v(x) = v_0(x), v_\lambda(x) = \frac{\lambda^{n-2}}{|x|^{n-2}} v(\frac{\lambda^2 x}{|x|^2}), w_\lambda(x) = v(x) - v_\lambda(x)$. Clearly, w_λ satisfies:

$$\begin{cases} -\Delta w_\lambda & = c_1(x) w_\lambda \qquad \text{in } B_\lambda^+, \\ \frac{\partial w_\lambda}{\partial t} & = c_2(x) w_\lambda \qquad \text{on } \partial B_\lambda^+ \cap \partial\mathbb{R}_+^n \setminus \{0\}, \end{cases} \tag{13}$$

where $c_1 = n(n+2)\xi_1(x)^{\frac{4}{n-2}}$, $c_2 = \frac{nc}{n-2}\xi_2(x)^{\frac{2}{n-2}}$, ξ_1, ξ_2 are two functions between v_λ and v.

Claim 1: For λ large enough, $w_\lambda(x) \geq 0$ for all $x \in \overline{B}_\lambda^+ \setminus \{0\}$.

Proof: We prove this claim by three steps.

Step 1: $\exists R_0 > 0$, such that for all $R_0 \leq |x| \leq \lambda/2$, we have $w_\lambda(x) \geq 0$.

Proof: Set $A = \lim_{|y|\to\infty} |y|^{n-2} v(y) = u(0)$. It is clear that for $R_0 > 0$ large, we have for all $R_0 \leq |x| \leq \lambda/2$ that

$$\begin{aligned} w_\lambda(x) &= \tfrac{1}{|x|^{n-2}}(|x|^{n-2}v(x)) - \tfrac{1}{\lambda^{n-2}}(|\tfrac{\lambda^2 x}{|x|^2}|^{n-2} v(\tfrac{\lambda^2 x}{|x|^2})) \\ &= \tfrac{1}{|x|^{n-2}}(A + O(\tfrac{1}{|x|})) - \tfrac{1}{\lambda^{n-2}}(A + O(\tfrac{|x|}{\lambda^2})) \\ &\geq \tfrac{1}{|x|^{n-2}}\{(1-\tfrac{1}{2})A - O(\tfrac{1}{R_0})\} \\ &> 0. \end{aligned}$$

Step 1 is established.

Step 2: $\exists R_1 \geq R_0$, such that for $R_1 \leq \lambda/2 \leq |x| \leq \lambda$, we have $w_\lambda(x) \geq 0$.

Proof: Let $g(x) = |x|^{-\alpha}$ with $0 < \alpha < n-2$, and $\bar{w}_\lambda(x) = w_\lambda(x)/g(x)$. It follows from (13) that

$$\left\{ \begin{array}{ll} \Delta \bar{w}_\lambda & + \frac{2}{g}\nabla g \cdot \nabla \bar{w}_\lambda + (c_1(x) + \frac{\Delta g}{g})\bar{w}_\lambda = 0 \quad in \quad B_\lambda^+, \\ \frac{\partial \bar{w}_\lambda}{\partial t} & = c_2(x)\bar{w}_\lambda \quad on \quad \partial B_\lambda^+ \cap \partial \mathbb{R}_+^n \setminus \{0\}. \end{array} \right. \tag{14}$$

Suppose the contrary, $\exists\, x_0 = (x_0', t_0)$ with $\lambda/2 \leq |x_0| \leq \lambda$, such that $\bar{w}_\lambda(x_0) = \min\limits_{\lambda/2 \leq |x| \leq \lambda} \bar{w}_\lambda(x) < 0$. Then $|x_0| \neq \lambda$ from the definition of w_λ, $|x_0| \neq \lambda/2$ from Step 1. It follows that $\lambda/2 < |x_0| < \lambda$ and

$$v_\lambda(x_0) \leq 2^{n-2} v(\frac{\lambda^2 x_0}{|x_0|^2}) \leq \frac{C}{|x_0|^{n-2}},$$

$$v(x_0) \leq v_\lambda(x_0) \leq \frac{C}{|x_0|^{n-2}}.$$

Here and in the following, C denotes various constant independent of λ. Thus, $c_1(x_0) \leq \frac{C}{|x_0|^4}$. By a direct calculation we have

$$\frac{\Delta g}{g}(x) = -\frac{\alpha(n-2-\alpha)}{|x|^2}.$$

It is clear that for $R_1 \geq R_0$ large enough we have

$$c_1(x_0) + \frac{\Delta g}{g}(x_0) < 0. \tag{15}$$

For $c > 0$, it is clear that $t_0 > 0$. When $c = 0$ it follows from (15) and the strong form of the Hopf lemma ([25]) that $t_0 > 0$. Using (15), we reach a contradiction simply by evaluating (14) at x_0. Step 2 is established.

Step 3: $\exists\, R_2 \geq R_1$, such that for $\lambda \geq R_2$,

$$w_\lambda(x) \geq 0 \quad \text{for} \quad x \in B^+_{R_0} \setminus \{0\}.$$

Proof: It follows from Corollary 2.1 that $v(x) > 1/C > 0$ for $x \in B^+_{R_0} \setminus \{0\}$. Writing

$$w_\lambda(x) = v(x) - \frac{1}{\lambda^{n-2}}(|\frac{\lambda^2 x}{|x|^2}|^{n-2} v(\frac{\lambda^2 x}{|x|^2})).$$

Step 3 follows from the fact that $\lim_{|y|\to\infty} |y|^{n-2} v(y) = u(0)$ and $|\frac{\lambda^2 x}{|x|^2}| \geq \lambda^2/R_0 \geq R_2^2/R_0$ for all $x \in B^+_{R_0} \setminus \{0\}$.

We have verified Claim 1.

Now we define for $b \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$ that

$$\lambda_b = \inf\,\{\lambda > 0 \mid w_{\mu,b}(x) \geq 0 \quad \text{in} \quad \bar{B}^+_\mu \setminus \{0\} \quad \text{for all} \quad \lambda < \mu < \infty\}. \tag{16}$$

Claim 2: There exists $\bar{b} \in \mathbb{R}^{n-1}$, such that $\lambda_{\bar{b}} > 0$.
First we prove a general lemma.
Lemma 2.2: Suppose $f \in C^1(\mathbb{R}^n_+)$ satisfying: for all $b \in \mathbb{R}^{n-1}, \lambda > 0$,

$$f_b(x) - \frac{\lambda^{n-2}}{|x|^{n-2}} f_b(\frac{\lambda^2 x}{|x|^2}) \leq 0,\ \forall\, x \in B^+_\lambda,$$

where $f_b(x) = f(x' + b, t)$, $\forall x = (x', t) \in \mathbb{R}^n_+$. Then

$$f(x) = f(x', t) = f(0, t), \qquad \forall\, x = (x', t) \in \mathbb{R}^n_+.$$

Proof: For all $b \in \mathbb{R}^{n-1}$, $\lambda > 0$, set

$$g_{b,\lambda}(x) \equiv f_b(x) - \frac{\lambda^{n-2}}{|x|^{n-2}} f_b(\frac{\lambda^2 x}{|x|^2}),\ x \in \mathbb{R}^n_+, |x| \leq \lambda.$$

It is easy to see that for all $b \in \mathbb{R}^{n-1}$, $x \in \mathbb{R}^n_+$, we have

$$\begin{cases} g_{b,|x|}(x) & = 0, \\ g_{b,|x|}(rx) & \leq 0,\ \forall\, 0 < r < 1. \end{cases}$$

It follows that

$$\frac{d}{dr}\{g_{b,|x|}(rx)\}|_{r=1} \geq 0.$$

A direct computation yields

$$2\nabla f_b(x) \cdot x + (n-2) f_b(x) \leq 0,$$

namely,

$$2\partial_{x'} f(x'+b,t) \cdot x' + 2\partial_t f(x'+b,t)t + (n-2)f(x'+b,t) \geq 0.$$

Since $x', b \in \mathbb{R}^{n-1}$, $t > 0$ are arbitrary, by a change of variable we have

$$2\partial_{x'} f(x',t) \cdot (x'-b) + 2\partial_t f(x',t)t + (n-2)f(x',t) \geq 0.$$

Dividing the above by $|b|$ and sending $|b|$ to infinity, we have: for all $x' \in \mathbb{R}^{n-1}$, $t > 0$, $\omega \in \mathbb{R}^{n-1}$, $|\omega| = 1$,

$$-\partial_{x'} f(x',t) \cdot \omega \geq 0.$$

It follows that

$$\partial_{x'} f(x',t) = 0, \ \forall x' \in \mathbb{R}^{n-1}, \ t > 0.$$

Lemma 2.2 is established.

Proof of Claim 2: If $\lambda_b = 0$ for all $b \in \mathbb{R}^{n-1}$, we have for all $b \in \mathbb{R}^{n-1}$ and $\lambda > 0$

$$v_b(x) \geq \frac{\lambda^{n-2}}{|x|^{n-2}} v_b(\frac{\lambda^2 x}{|x|^2}) \qquad \forall x \in B^+_\lambda.$$

It follows that for all $b \in \mathbb{R}^{n-1}$, and $\lambda > 0$ that

$$u_b(y) \leq \frac{\lambda^{2-n}}{|y|^{n-2}} u_b(\frac{\lambda^{-2} y}{|y|^2}) \qquad \forall y \in B^+_{1/\lambda}.$$

Therefore, from Lemma 2.2 we know that $u(x)$ depends only on t. Writing $u(t) = u(x)$, it follows that $u \in C^2(0,\infty) \cap C^1[0,\infty)$ satisfies

$$\begin{cases} u''(t) + n(n-2)u^{\frac{n+2}{n-2}}(t) & = 0, \quad 0 < t < \infty, \\ u(t) & > 0, \quad 0 \leq t < \infty. \end{cases} \tag{17}$$

An elementary phase-plane argument (writing (17) as a first order autonomous system) shows that (17) has no solution. This is a contradiction. We have verified Claim 2.

Claim 3: Suppose $\lambda_b > 0$ for some $b \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$, then we have $w_{\lambda_b,b}(x) = 0, \ \forall\ x \in \mathbb{R}^n_+$.

Proof: Without loss of generality, we assume $b = 0$. Suppose the contrary of Claim 3, w_{λ_0} satisfies

$$\begin{cases} \Delta w_{\lambda_0} & \leq 0 & \text{in } \ B^+_{\lambda_0}(0), \\ \frac{\partial w_{\lambda_0}}{\partial t} & = cv^{\frac{n}{n-2}} - cv_{\lambda_0}^{\frac{n}{n-2}} & \text{on } \ \partial B^+_{\lambda_0} \cap \partial\mathbb{R}^n_+ \setminus \{0\}, \\ w_{\lambda_0} & \geq 0, & \text{in } \ \bar{B}^+_{\lambda_0} \setminus \{0\}. \end{cases} \tag{18}$$

It follows from the strong form of the maximum principle and the Hopf lemma (using also the boundary condition of w_{λ_0} given in (18)) that

$$\begin{cases} w_{\lambda_0}(x) > 0, & \forall\ x \in \bar{B}^+_{\lambda_0},\ 0 < |x| < \lambda_0, \\ \frac{\partial w_{\lambda_0}}{\partial \nu}(x) > 0, & \forall\ x \in \partial B^+_{\lambda_0} \cap \mathbb{R}^n_+, \end{cases} \tag{19}$$

where ν denotes the inner unit normal of the sphere ∂B_{λ_0}.

Lemma 2.3: There exists some constant $\gamma = \gamma(\lambda_0) > 0$ such that $w_{\lambda_0}(x) \geq \gamma$ for $x \in \overline{B}^+_{\lambda_0/2} \setminus \{0\}$.

Proof: Recall that $v_{\lambda_0}(x) = \frac{\lambda_0^{n-2}}{|x|^{n-2}} v(\frac{\lambda_0^2 x}{|x|^2})$, $v_{\lambda_0}(x) \to {\lambda_0}^{2-n} u(0)$ as $|x| \to 0$. Thus $\exists C_1 > 0$ such that $v_{\lambda_0}(x) < C_1$ for $|x| < \lambda_0$. Using (19), we have $\min\limits_{\partial B^+_{\lambda_0/2} \cap \partial B_{\lambda_0/2}} w_{\lambda_0} \geq \epsilon$, for some $0 < \epsilon < 1$. Without loss of generality, we assume $\lambda_0 = 2$. For $0 < r < 1$, we introduce an auxilary function,

$$\varphi_1(x) = \frac{\epsilon\mu}{2(c+1)} - \frac{r^{n-2}\epsilon}{|x|^{n-2}} + \frac{\epsilon t(1-\mu)}{2}, \qquad x \in B^+_1 \setminus B^+_r, \tag{20}$$

where $0 < \mu < 1$ being chosen later. Let $P(x) = w_{\lambda_0}(x) - \varphi_1(x)$, then $P(x)$ satisfies

$$\begin{cases} \Delta P(x) & \leq 0 & \text{in } \ B^+_1 \setminus \bar{B}^+_r, \\ \frac{\partial P(x)}{\partial t} & = cv^{\frac{n}{n-2}} - cv_{\lambda_0}^{\frac{n}{n-2}} - \frac{\epsilon(1-\mu)}{2} & \text{on } \ \partial(B^+_1 \setminus \bar{B}^+_r) \cap \partial\mathbb{R}^n_+. \end{cases} \tag{21}$$

We will show that

$$P(x) \geq 0, \qquad \forall\ x \in \bar{B}^+_1 \setminus B^+_r. \tag{22}$$

On $\partial B^+_1 \cap \partial B_1 : P(x) \geq \epsilon - (\epsilon - r^{n-2}\epsilon) > 0$; on $\partial B^+_r \cap \partial B_r : P(x) > w_{\lambda_0}(x) \geq 0$. If (22) does not hold, there exists some $x_0 = (x'_0, t_0)$ such that, $P(x_0) = \min\limits_{x \in \overline{B}^+_1 \setminus B^+_r} P(x) < 0$. It follows from the above consideration and the maximum principle that $t_0 = 0, r < |x_0'| < 1$. Then we have $\frac{\partial P(x)}{\partial t}(x_0) \geq 0$. From $P(x_0) < 0$, we have $v(x_0) - v_{\lambda_0}(x_0) - \varphi_1(x_0) < 0$. It follows that

$$v(x_0) < C_2, \tag{23}$$

for some constant $C_2 = C(\epsilon, C_1)$. Also

$$w_{\lambda_0}(x_0) < \frac{\epsilon\mu}{2(c+1)} - \frac{r^{n-2}\epsilon}{|x_0|^{n-2}} < \frac{\epsilon\mu}{2(c+1)}. \tag{24}$$

From (23),(24), and the mean value theorem we have $v^{\frac{n}{n-2}}(x_0) - v_{\lambda_0}^{\frac{n}{n-2}}(x_0) < C_3 w_{\lambda_0}(x_0)$ for some constant $C_3 > 0$. Combining with $\frac{\partial P}{\partial t}(x_0) \geq 0$, we have

$$w_{\lambda_0}(x_0) \geq \frac{\epsilon}{2(c+1)C_3} \cdot (1-\mu). \tag{25}$$

Combining (24)and (25) we have

$$\mu > \frac{1}{1+C_3}.$$

If we choose μ such that $0 < \mu < \frac{1}{1+C_3}$ from the beginning, We reach a contradiction. (22) is established. Let $r \to 0$, we have proved Lemma 2.3 with $\gamma = \frac{\epsilon}{2(c+1)(C_3+1)}$.

From the definition of λ_0, there exists a sequence $\lambda_k \to \lambda_0$ with $\lambda_k < \lambda_0$, such that

$$\inf_{\bar{B}^+_{\lambda_k} \setminus \{0\}} w_{\lambda_k} < 0.$$

It is not difficult to see from Lemma 2.3 and the continuity of u at 0 that for k large enough, we have

$$w_{\lambda_k}(x) \geq \gamma/2, \qquad \forall\, x \in \overline{B}^+_{\lambda_0/2} \setminus \{0\}.$$

It follows that there exists $x_k = (x_k', t_k) \in \bar{B}^+_{\lambda_k} \setminus B^+_{\lambda_0/2}$ such that

$$w_{\lambda_k}(x_k) = \min_{\bar{B}^+_{\lambda_k} \setminus \{0\}} w_{\lambda_k} < 0.$$

It is clear that $\lambda_0/2 < |x_k| < \lambda_k$ and, due to the boundary condition, $t_k > 0$. Hence $\nabla w_{\lambda_k}(x_k) = 0$. After passing to a subsequence (still denoted as x_k) $x_k \to x_0 = (x_0', t_0)$. It follows that

$$w_{\lambda_0}(x_0) = 0, \qquad \nabla w_{\lambda_0}(x_0) = 0. \tag{26}$$

It follows from (26) and (19) that $t_0 = 0, |x_0'| = \lambda_0$.

Lemma 2.4: If (18) and (19) hold, then $\frac{\partial w_{\lambda_0}}{\partial \nu}(x_0) > 0$ for all $x_0 = (x_0', 0)$, $|x_0'| = \lambda_0$.

Once we establish Lemma 2.4, we reach a contradiction due to (26), thus have verified Claim 3.

Proof of Lemma 2.4: Without loss of generality we assume $\lambda_0 = 1$. Set $\Omega = \{x = (x', t) \mid x \in B_1^+ \backslash \bar{B}_{1/2}^+,\ t < 1/4\}$ and, for some $\alpha > \max\{n/2, n-3\}$,

$$h(x) = \epsilon(|x'|^{-\alpha} - 1)(t + \mu),\ \varphi_2(x) = h(x) - \frac{1}{|x|^{n-2}} h(\frac{x}{|x|^2}), \quad x \in \Omega,$$

where $0 < \epsilon, \mu < 1$ being chosen later. A direct computation yields

$$\Delta \varphi_2(x) \geq 0, \qquad \forall\ x \in \Omega.$$

Consider $A(x) = w_{\lambda_0}(x) - \varphi_2(x)$, it follows that

$$\begin{cases} \Delta A & \leq 0, & \text{in } \ \Omega, \\ \frac{\partial A}{\partial t} & = c_2(x) w_{\lambda_0}(x) - \frac{\partial \varphi_2}{\partial t} & \text{on } \partial\Omega \cap \partial \mathbb{R}_+^n, \end{cases}$$

where $c_2(x) = \frac{nc}{n-2} \xi_2(x)^{\frac{2}{n-2}}$, ξ_2 is some function between v_{λ_0} and v.

For suitably chosen ϵ and μ, we want to show

$$A(x) = w_{\lambda_0}(x) - \varphi_2(x) \geq 0, \quad \forall\ x \in \ \Omega. \tag{27}$$

Using (19), we can choose $\epsilon_0 > 0$ small enough, such that for all $0 < \epsilon < \epsilon_0$, we have $A(x) \geq 0$ on $\partial\Omega \cap \{\partial B_{1/2} \cup \{t = 1/4\}\}$. Also from the construction of φ_2, we know $A(x) = 0$ on $\partial\Omega \cap \partial B_1$. Suppose the contrary of (27), there exists some $x_1 = (x_1', t_1) \in \bar{\Omega}$ such that

$$A(x_1) = \min_{\bar{\Omega}} A < 0. \tag{28}$$

From the above and the maximum principle, we have $t_1 = 0, 1/2 < |x_1'| < 1$. Thus

$$\frac{\partial A}{\partial t}(x_1) \geq 0. \tag{29}$$

A simple calculation yields

$$\frac{\partial \varphi_2}{\partial t}(x_1) = \epsilon(|x_1'|^{-\alpha} - 1)(|x_1'|^{-n+\alpha} + 1). \tag{30}$$

Combining (29) and (30) we have

$$c_2(x_1)w_{\lambda_0}(x_1) - \epsilon(|x_1'|^{-\alpha} - 1)(|x_1'|^{-n+\alpha} + 1) \geq 0. \tag{31}$$

It follows from (28) that

$$w_{\lambda_0}(x_1) < \epsilon\mu(|x_1'|^{-\alpha} - 1)(|x_1'|^{-n+\alpha+2} + 1). \tag{32}$$

Combining (31) and (32) we have

$$c_2(x_1)\mu > 1.$$

So if we choose $0 < \mu < \min_{1/2 \leq |x| \leq 1}(c_2(x) + 1)^{-1}$ from the beginning, we reach a contradiction. Thus (27) holds. Since we also know that $A(x_0) = 0$, we have

$$\frac{\partial A}{\partial \nu}(x_0) \geq 0.$$

It follows from a direct computation that

$$\frac{\partial w_{\lambda_0}}{\partial \nu}(x_0) = \frac{\partial A}{\partial \nu}(x_0) + \frac{\partial \varphi_2}{\partial \nu}(x_0) \geq \frac{\partial \varphi_2}{\partial \nu}(x_0) = 2\alpha\epsilon\mu > 0$$

Lemma 2.4 is established, also Claim 3 as remarked earlier.

Claim 4: For all $b \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$, we have $\lambda_b > 0$.

Proof: It follows from Claim 2 and Claim 3 that there exists some $\bar{b} \in \mathbb{R}^{n-1}$ such that $\lambda_{\bar{b}} > 0$ and $w_{\lambda_{\bar{b}},\bar{b}}(x) = 0,\ \forall\ x \in \mathbb{R}^n_+$. It follows that

$$u_{\bar{b}}(y) = \lambda_{\bar{b}}^{2-n}|y|^{2-n}u_{\bar{b}}(\frac{y}{\lambda_{\bar{b}}^2|y|^2}), \quad \forall\ y \in \mathbb{R}^n_+.$$

Clearly

$$\lim_{|y|\to\infty} |y|^{n-2}u_{\bar{b}}(y) = \lambda_{\bar{b}}^{2-n}u(\bar{b}, 0),$$

namely,

$$\lim_{|y|\to\infty} |y|^{n-2}u(y) = \lambda_{\bar{b}}^{2-n}u(\bar{b}, 0). \tag{33}$$

Suppose the contrary of Claim 4 for some $b \in \mathbb{R}^{n-1}$, namely,

$$w_{\lambda,b}(x) = v_b(x) - \frac{\lambda^{n-2}}{|x|^{n-2}}v_b(\frac{\lambda^2 x}{|x|^2}) \geq 0, \ \ \forall\ \lambda > 0,\ x \in \bar{B}^+_\lambda \setminus \{0\}.$$

It follows that

$$\frac{\lambda^{n-2}}{|x|^{n-2}} u_b(\frac{x}{|x|^2}) \geq u_b(\frac{x}{\lambda^2}), \quad \forall\ \lambda > 0,\ x \in \bar{B}^+_\lambda \setminus \{0\}.$$

Fixing $\lambda > 0$ in the above and sending $|x|$ to 0, we have (using (33))

$$\frac{\lambda^{n-2}}{\lambda_{\bar{b}}^{n-2}} u(\bar{b}, 0) \geq u_b(0).$$

Sending λ to 0, we have

$$u(b, 0) = u_b(0) \leq 0,$$

a contradiction. Claim 4 has been verified.

Proposition 2.1 follows immediately from Claim 1-4.

Lemma 2.5: Suppose $f \in C^1(\mathbb{R}^{n-1})$ $(n \geq 3)$ satisfying: $\forall\ b \in \mathbb{R}^{n-1}$, there exists $\mu_b \in \mathbb{R}$ such that

$$f(x' + b) = \frac{\mu_b^{n-2}}{|x'|^{n-2}} f(\frac{\mu_b^2 x'}{|x'|^2} + b), \quad \forall\ x' \in \mathbb{R}^{n-1} \setminus \{0\}. \tag{34}$$

Then for some $a \geq 0, d > 0, x_0' \in \mathbb{R}^{n-1}$,

$$f(x') = (\frac{a}{|x' - x_0'|^2 + d})^{(n-2)/2}, \quad \forall\ x' \in \mathbb{R}^{n-1},$$

or

$$f(x') = -(\frac{a}{|x' - x_0'|^2 + d})^{(n-2)/2}, \quad \forall\ x' \in \mathbb{R}^{n-1}.$$

Proof: Rewriting it as

$$f(x') = \frac{\mu_b^{n-2}}{|x' - b|^{n-2}} f(\frac{\mu_b^2 (x' - b)}{|x' - b|^2} + b), \quad \forall\ x' \in \mathbb{R}^{n-1} \setminus \{b\}.$$

It follows that

$$A := \lim_{|x'| \to \infty} |x'|^{n-2} f(x') = \mu_b^{n-2} f(b), \quad \forall\ b \in \mathbb{R}^{n-1}. \tag{35}$$

If $A = 0$, it is easy to see from (34) and (35) that $f \equiv 0$. If $A \neq 0$, both $f(b)$ and μ_b can not change sign. Without loss of generality we assume that $A = 1$. It follows from (35) that $f(b) > 0, \mu_b > 0, \forall\ b \in \mathbb{R}^{n-1}$.

For x' large,

$$f(x') = \frac{{\mu_0}^{n-2}}{|x'|^{n-2}}\{f(0) + \frac{\partial f}{\partial x_i'}(0) \cdot \frac{{\mu_0}^2 x_i'}{|x'|^2} + \circ(\frac{1}{|x'|})\}, \tag{36}$$

and

$$f(x') = \frac{{\mu_b}^{n-2}}{|x'-b|^{n-2}}\{f(b) + \frac{\partial f}{\partial x_i'}(b) \cdot \frac{{\mu_b}^2 (x_i'-b_i)}{|x'-b|^2} + \circ(\frac{1}{|x'|})\}, \tag{37}$$

Combining (36), (37), (35) and our assumption $A = 1$ we have

$$f^{-\frac{n}{n-2}}(b) \cdot \frac{\partial f}{\partial x_i'}(b) = f^{-\frac{n}{n-2}}(0) \cdot \frac{\partial f}{\partial x_i'}(0) - (n-2)b_i,$$

It follows that for some $x_0' \in \mathbb{R}^{n-1}$, $d > 0$

$$f^{-\frac{2}{n-2}}(x') = |x' - x_0'|^2 + d.$$

Lemma 2.6: For some $a, d > 0$, $x_0' \in \mathbb{R}^{n-1}$,

$$u(x', 0) = (\frac{a}{|x'-x_0'|^2 + d})^{(n-2)/2}, \quad \forall\ x' \in \mathbb{R}^{n-1}.$$

Proof : It follows from Lemma 2.5 and $u > 0$ in $\mathbb{R}^n$.
Proof of Theorem 1.1 for $c \geq 0$: Let $x_0 = (x_0', -\sqrt{d})$, set

$$\varphi(x) = \frac{1}{|x-x_0|^{n-2}} u(\frac{x-x_0}{|x-x_0|^2} + x_0),$$

and $B = \{(x', t) : \frac{t+\sqrt{d}}{|x'-x_0'|^2 + (t+\sqrt{d})^2} - \sqrt{d} > 0\}$. Clearly, B is a ball in $\mathbb{R}^n$. Without loss of generality, we assume $a = 1$ in Lemma 2.6. It follows that on ∂B

$$\begin{aligned} \varphi(x', t) &= [|x'-x_0'|^2 + (t+\sqrt{d})^2]^{\frac{2-n}{2}} [d + \tfrac{|x'-x_0'|^2}{(|x'-x_0'|^2 + (t+\sqrt{d})^2)^2}]^{\frac{2-n}{2}} \\ &= \{d[|x'-x_0'|^2 + (t+\sqrt{d})^2] + \tfrac{|x'-x_0'|^2}{|x'-x_0'|^2 + (t+\sqrt{d})^2}\}^{\frac{2-n}{2}} \\ &= \{\tfrac{d(t+\sqrt{d})}{\sqrt{d}} + \tfrac{\sqrt{d}|x'-x_0'|^2}{t+\sqrt{d}}\}^{\frac{2-n}{2}} = 1. \end{aligned}$$

Define $Q(x) = \varphi(x) - 1$, we know that $Q = 0$ on ∂B and

$$-\Delta Q = n(n-2)(Q+1)^{\frac{n+2}{n-2}} \quad in \ \ B. \tag{38}$$

It follows from the maximum principle that $Q > 0$ in B. Applying the result of [15] we know that Q is radially symmetric about the center of B. Hence by the uniqueness of the ode solution of (38), $\varphi(x)$ must take the form $\varphi(x) = (\frac{\epsilon}{\epsilon^2+|x-x_1|^2})^{(n-2)/2}$ for some $\epsilon > 0$ and $x_1 \in \mathbb{R}^n$. Then

$$\begin{aligned} u(y) &= \tfrac{1}{|y-x_0|^{n-2}}\varphi(\tfrac{y-x_0}{|y-x_0|^2}+x_0) \\ &= (\tfrac{\epsilon}{(\epsilon^2+|x_0-x_1|^2)|y-x_0|^2+2(y-x_0)(x_0-x_1)+1})^{\frac{n-2}{2}}. \end{aligned}$$

Theorem 1.1 when $c \geq 0$ follows immediately.

2.2 Case $c < 0$.

The main procedure of this case is similar to Case $c \geq 0$, so we define $u_b, v(x)$,$v_b(x)$,$w_{\lambda,b}$ and w_λ as before. It is clear that this case is much easier than the case $C > 0$ since the boundary condition has the good sign.

Proposition 2.2 Let u be a positive function satisfying the hypotheses of Theorem 1.1 for some $c < 0$, then for any $b \in \mathbb{R}^{n-1}$, there exists a $\lambda_b > 0$ such that

$$u_b(x) = \frac{\lambda_b^{2-n}}{|x|^{n-2}}u_b(\frac{\lambda_b^{-2}x}{|x|^2}) \quad for \quad x \in \mathbb{R}^n_+.$$

Lemma 2.7: Let $v \in C^2(\mathbb{R}^n_+) \cap C^1(\bar{\mathbb{R}}^n_+) \backslash \{0\}$ satisfy (10), $\epsilon = \min\limits_{\partial B_R^+ \cap \partial B_R} v(x)$, $R > 0$. Then we have $v(x) \geq \epsilon$ for all $x \in \overline{B}_R^+ \setminus \{0\}$.

Proof: Since $c < 0$, it follows from the maximum principle that $v(x) \geq \epsilon - \epsilon\frac{r^{n-2}}{|x|^{n-2}}$, $\forall\ 0 < r < R, x \in \overline{B}_R^+ \setminus B_r^+$. The lemma follows by sending r to 0.

Proof of Proposition 2.2: We still prove it by four Claims. As in Case $c \geq 0$, w_λ satisfies (13).

Claim 1: For λ large enough, $w_\lambda(x) \geq 0$ for all $x \in \overline{B}_\lambda^+ \setminus \{0\}$.

Proof: The proof is still divided into three steps.

Step 1: $\exists R_0 > 0$, such that for all $R_0 \leq |x| \leq \lambda/2$, we have $w_\lambda(x) \geq 0$.

Proof: The same as in Case $c \geq 0$.

Step 2: $\exists R_1 \geq R_0$, such that for $R_1 \leq \lambda/2 \leq |x| \leq \lambda$, we have $w_\lambda(x) \geq 0$.

Proof: Let $g(x) = |z|^{-\alpha}$ with $0 < \alpha < n-2$, $z = x + (0, 0, ..., \frac{\lambda}{4})$. Also $\bar{w}_\lambda(x) = w_\lambda(x)/g(x)$.

$$\begin{cases} \Delta\bar{w}_\lambda & +\frac{2}{g}\nabla g \cdot \nabla\bar{w}_\lambda + (c_1(x) + \frac{\Delta g}{g})\bar{w}_\lambda = 0 \qquad \text{in} \quad B_\lambda^+, \\ \frac{\partial\bar{w}_\lambda}{\partial t} & = (c_2(x) - \frac{1}{g}\cdot\frac{\partial g}{\partial t})\bar{w}_\lambda \quad \text{on} \quad \partial B_\lambda^+ \cap \partial\mathbb{R}^n_+ \setminus \{0\}. \end{cases} \tag{39}$$

If $\exists x_0$ with $\lambda/2 \le |x_0| \le \lambda$, such that $\bar{w}_\lambda(x_0) = \min\limits_{\lambda/2 \le |x_0| \le \lambda} \bar{w}_\lambda(x) < 0$. Then $x_0 \notin \{|x| = \lambda\} \cup \{|x| = \lambda/2\}$. As in Case $c \ge 0$,

$$0 \le c_1(x_0) \le \frac{C_1}{|x_0|^4}, \quad 0 \le -c_2(x_0) \le \frac{C_2}{|x_0|^2}.$$

Also $\frac{\Delta g}{g} = -\frac{\alpha(n-2-\alpha)}{|z|^2}$, $-\frac{1}{g} \cdot \frac{\partial g}{\partial t}|_{t=0} = \frac{\alpha\lambda}{4|z|^2}$. For λ large enough, $z_0 = x_0 + (0, 0, ..., \frac{\lambda}{4})$,

$$-\frac{\alpha(n-2-\alpha)}{|z_0|^2} + c_1(x_0) < 0,$$

and

$$\frac{\alpha\lambda}{4|z_0|^2} + c_2(x_0) > 0, \quad \text{when } x_0 \in \partial\mathbb{R}^n_+.$$

This contradicts to the maximum principle.

Step 3: $\exists\ R_2 \ge R_1$, such that for $\lambda \ge R_2$,

$$w_\lambda(x) \ge 0 \ \text{ for } \ x \in B^+_{R_0} \setminus \{0\}.$$

Proof: From Lemma 2.7 we have $v(x) > 1/C > 0$ for $x \in B^+_{R_0} \setminus \{0\}$. Step 3 follows as in Case $c \ge 0$.

We have verified Claim 1.

Define λ_b for $b \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$ as in (16).

Claim 2: There exists $\bar{b} \in \mathbb{R}^{n-1}$, such that $\lambda_{\bar{b}} > 0$.

Proof of Claim 2: If $\lambda_b = 0$ for all $b \in \mathbb{R}^{n-1}$, it follows from Lemma 2.2 that $u(x)$ just depends on t. Writing $u(t) = u(x)$, it follows that $u \in C^2(0, \infty) \cap C^1[0, \infty)$ satisfies (17). Claim 2 follows exactly as in Case $c \ge 0$.

Claim 3: Suppose $\lambda_b > 0$ for some $b \in \partial\mathbb{R}^n_+ = \mathbb{R}^{n-1}$, then we have $w_{\lambda_b, b}(x) = 0, \ \forall\ x \in \mathbb{R}^n_+$.

Proof: Without loss of generality, we assume $b = 0$. Suppose the contrary of Claim 3, w_{λ_0} satisfies (18) and (19) with $c < 0$. It follows that

$$\frac{\partial w_{\lambda_0}}{\partial t}(x) < 0, \quad x \in \partial\mathbb{R}^n_+, \ 0 < |x| < \lambda_0. \tag{40}$$

Lemma 2.8: There exists some constant $\gamma = \gamma(\lambda_0) > 0$ such that $w_{\lambda_0}(x) \ge \gamma$ for $x \in \overline{B}^+_{\lambda_0/2} \setminus \{0\}$.

Proof: Set $\gamma = \min\limits_{\partial B^+_{\lambda_0/2} \cap \partial B_{\lambda_0/2}} w_{\lambda_0}(x)$. It follows from (19) that $\gamma > 0$. Using (18) , (40) and the maximum principle we have $w_{\lambda_0}(x) \ge \gamma -$

$\gamma\frac{r^{n-2}}{|x|^{n-2}}, \forall\, 0 < r < \lambda_0/2, x \in \bar{B}^+_{\lambda_0/2} \setminus B^+_r$. Lemma 2.8 follows after sending r to 0.

From the definition of λ_0 and arguing in a way similar to that in Case $c \geq 0$, there exists some $x_0 = (x_0', 0)$ with $x_0' \in \mathbb{R}^{n-1}$, $|x_0'| = \lambda_0$, such that

$$w_{\lambda_0}(x_0) = 0, \qquad \frac{\partial w_{\lambda_0}}{\partial \nu}(x_0) = 0,$$

where ν denotes the inner unit normal of ∂B_{λ_0}.

Lemma 2.9: If (18) and (19) hold, then $\frac{\partial w_{\lambda_0}}{\partial \nu}(x_0) > 0$ for all $x_0 = (x_0', 0)$, $|x_0'| = \lambda_0$.

Proof: Without loss of generality we assume $\lambda_0 = 1$. Set $\Omega = \{x = (x', t) \mid x \in B_1^+ \setminus \bar{B}^+_{1/2},\ t < 1/4\}$ and, for some $\alpha > \max\{n/2, n-3\}$,

$$h(x) = \epsilon(|x'|^{-\alpha} - 1)(1 + t), \quad \varphi_3(x) = h(x) - \frac{1}{|x|^{n-2}} h(\frac{x}{|x|^2}),$$

where $\epsilon > 0$ will be chosen later. A direct computation yields

$$\Delta\varphi_3(x) \geq 0, \qquad x \in \Omega.$$

Considering $A(x) = w_{\lambda_0}(x) - \varphi_3(x)$, it satisfies

$$\begin{cases} \Delta A & \leq 0, \quad \text{in } \ \Omega, \\ \frac{\partial A}{\partial t} & \leq 0 \quad \ \text{on } \partial\Omega \cap \{t = 0\}. \end{cases} \tag{41}$$

By (19) we can choose ϵ_0 small enough, such that for $0 < \epsilon < \epsilon_0$, we have $A(x) > 0$ on $\partial\Omega \cap \{\partial B_{1/2} \cup \{t = 1/4\}\}$. Also from the construction of φ_3 we know $A(x) = 0$ on $\partial\Omega \cap \partial B_1$. It follows from the maximum principle that

$$A \geq 0, \quad \text{in } \ \Omega. \tag{42}$$

Also we know $A(x_0) = 0$, thus $\frac{\partial A}{\partial \nu}(x_0) \geq 0$. A direct computation yields

$$\frac{\partial \varphi_3}{\partial \nu}(x_0) = 2\alpha\epsilon.$$

It follows that

$$\frac{\partial w_{\lambda_0}}{\partial \nu}(x_0) \geq \frac{\partial \varphi_3}{\partial \nu}(x_0) > 0.$$

Lemma 2.9 is established and Claim 3 follows immediately.

Claim 4: For all $b \in \partial \mathbb{R}^n_+ = \mathbb{R}^{n-1}$, we have $\lambda_b > 0$.
Proof: It is exactly the same as that in Case $c \geq 0$.
Proposition 2.2 follows from Claim 1-4.

The rest of the proof of Theorem 1.1 in Case $c < 0$ follows immediately the same way as in Case $c \geq 0$.

3 Proof of Theorem 1.3

Proposition 3.1: Suppose $u \in C^2(\mathbb{R}^2_+) \cap C^1(\bar{\mathbb{R}}^2_+)$ satisfies (5) and (7). Then $\sup_{\mathbb{R}^2_+} u < +\infty$.

Lemma 3.1: Suppose $u \in C^2(B_3^+) \cap C^1(\bar{B}_3^+)$ satisfies, for some $c, A_1 \in \mathbb{R}$,

$$\begin{cases} -\Delta u &= e^u, & \text{in } \ B_3^+, \\ \frac{\partial u}{\partial t} &= c e^{u/2}, & \text{on } \ \{t=0\} \cap \bar{B}_3^+, \\ u(x_0) &= 1, & \text{for some } x_0 \in \ \bar{B}_1^+, \\ u &\leq A_1, & \text{in } \ B_3^+. \end{cases} \tag{43}$$

Then there exists some constant $C_1 = C_1(c, A_1)$, such that, $u(x) \geq -C_1$ in $\bar{B}_1^+$.

Proof: Set

$$\Gamma_1 = \{t = 0\} \cap \bar{B}_2^+, \quad \Gamma_2 = \{t > 0\} \cap \partial B_2^+.$$

Let

$$u_2(y) = \frac{c}{2\pi} \int_{\partial \mathbb{R}^2_+ \cap \partial B_3^+} (\log|x - y| + \log|x - \bar{y}|) e^{u(x)/2} dx, \quad y \in \bar{B}_2^+,$$

where $\bar{y}$ is the reflection point of y about $\{t = 0\}$.

A direct computation yields

$$\begin{cases} -\Delta u_2 &= 0 & \text{in } \ B_2^+, \\ \frac{\partial u_2}{\partial t} &= c e^{u/2} & \text{on } \ \Gamma_1. \end{cases}$$

Define u_1, u_3 by

$$\begin{cases} -\Delta u_1 &= e^u, & \text{in } \ B_2^+, \\ \frac{\partial u_1}{\partial t} &= 0, & \text{on } \ \Gamma_1, \\ u_1 &= 0, & \text{on } \ \Gamma_2, \end{cases}$$

$$\begin{cases} -\Delta u_3 &= 0, & \text{in } B_2^+, \\ \frac{\partial u_3}{\partial t} &= 0, & \text{on } \Gamma_1, \\ u_3 &= u - u_2, & \text{on } \Gamma_2. \end{cases}$$

Clearly $u = u_1 + u_2 + u_3$.

Extending u_1 evenly, we have

$$\begin{cases} -\Delta u_1 &= e^u, & \text{in } B_2, \\ u_1 &= 0, & \text{on } \partial B_2. \end{cases}$$

Since $e^u \leq e^{A_1}$, it follows from the $W^{2,p}$ estimates that

$$\max_{\bar{B}_2} |u_1| \leq C(A_1, c). \tag{44}$$

A direct computation yields for $y \in B_2^+$

$$\begin{aligned} |u_2(y)| &= \tfrac{|c|}{2\pi} \left| \int_{\partial \mathbb{R}^2_+ \cap \partial B_3^+} (\log|x-y| + \log|x-\bar{y}|) e^{u(x)/2} dx \right| \\ &\leq \tfrac{|c|}{2\pi} e^{A_1/2} \int_{\partial \mathbb{R}^2_+ \cap \partial B_3^+} |\log|x-y| + \log|x-\bar{y}|| dx \\ &\leq C(A_1, c). \end{aligned} \tag{45}$$

Reflecting u_3 evenly, we have

$$\begin{cases} -\Delta u_3 &= 0, & \text{in } B_2, \\ u_3 &= u - u_2, & \text{on } \partial B_2. \end{cases}$$

Notice that $u = u_1 + u_2 + u_3$, we have

$$u_3(x) = u(x) - u_1(x) - u_2(x) \leq A_1 + C(A_1, c), \quad \textit{on} \quad B_2^+,$$

and

$$u_3(x_0) = u(x_0) - u_1(x_0) - u_2(x_0) \geq 1 - C(A_1, c).$$

Applying the Harnack inequality to $A_1 + C(A_1, c) - u_3$, we have

$$\min_{B_1^+} u_3(x) \geq -C(c, A_1). \tag{46}$$

Lemma 3.1 follows from (44),(45),(46).

Lemma 3.2: For $c \in \mathbb{R}$, there exists $\epsilon_0 = \epsilon_0(c) > 0$, such that for all $u \in C^2(B_2^+) \cap C^1(\bar{B}_2^+)$ satisfying

$$\begin{cases} -\Delta u = & e^u, & \text{in } B_2^+, \\ \frac{\partial u}{\partial t} = & ce^{u/2} & \text{on } \partial B_2 \cap \{t=0\}, \\ \int_{B_2^+} e^u < & \epsilon_0, & \end{cases} \tag{47}$$

we have

$$\max_{\bar{B}_{1/4}^+} u \leq C(c).$$

Proof: We prove it by contradiction through a blow up argument used in the proof of Proposition 2.1 of [20]. Suppose the contrary, there exists some u_i satisfying (47), $x_i \in \bar{B}_{1/4}^+$, such that $u_i(x_i) \to \infty$.

In the following, we always denote $S^T = S \cap \{t > T\}$ for any set S and $S^+ = S^0$. Considering $(\frac{1}{2} - |x - x_i|)^2 e^{u_i(x)}$ in $B_{1/2}^+(x_i)$, there exist $y_i = (s_i, t_i) \in \overline{B}_{1/2}^+(x_i)$, such that

$$(\frac{1}{2} - |y_i - x_i|)^2 e^{u_i(y_i)} = \max_{x \in \overline{B}_{1/2}^+(x_i)} (\frac{1}{2} - |x - x_i|)^2 e^{u_i(x)}.$$

Let $\sigma_i = \frac{1}{2}(\frac{1}{2} - |y_i - x_i|) > 0$. We have

$$4\sigma_i^2 e^{u_i(y_i)} = \max_{x \in \overline{B}_{1/2}^+(x_i)} (\frac{1}{2} - |x - x_i|)^2 e^{u_i(x)} \geq \frac{1}{4} e^{u_i(x_i)}.$$

Thus

$$u_i(y_i) + 2\log\sigma_i \geq u_i(x_i) - 2\log 4 \to \infty. \tag{48}$$

Also for $x \in \overline{B}_{\sigma_i}^+(y_i)$, $\frac{1}{2} - |x - x_i| \geq \sigma_i$. Therefore

$$4\sigma_i^2 e^{u_i(y_i)} \geq \sigma_i^2 \max_{\overline{B}_{\sigma_i}^+(y_i)} e^{u_i},$$

namely,

$$u_i(y_i) \geq \max_{\bar{B}_{\sigma_i}^+(y_i)} u_i - \log 4.$$

Considering

$$w_i(x) = u_i(\mu_i x + y_i) + 2\log\mu_i,$$

with $\mu_i = 2e^{-\frac{u_i(y_i)}{2}}$. Then

$$\begin{cases} -\Delta w_i = & e^{w_i} \quad \text{in } B_{R_i}^{T_i}, \\ \frac{\partial w_i}{\partial t} = & ce^{w_i/2} \quad \text{on } \partial B_{R_i}^{T_i} \cap \{t = T_i\}, \\ \int_{B_{R_i}^{T_i}} e^{w_i} < & \epsilon_0, \\ w_i(x) \leq & 4\log 2 \quad for\ x \in \bar{B}_{R_i}^{T_i}, \\ w_i(0) = & 2\log 2, \end{cases}$$

where $R_i = \frac{\sigma_i}{\mu_i}, T_i = -t_i/\mu_i \leq 0$. We know from (48) that $R_i \to \infty$. Using Lemma 3.1 when $T_i \geq -1/2$ and Theorem 3 of [4] when $T_i < -1/2$, we conclude that for some constant $C = C(c)$ depending only on c , $w_i(x) > -C(c)$, $\forall\ x \in \overline{B}^+_{1/4}$. It follows that

$$\int_{B^+_{1/4}} e^{w_i(x)} > 1/C(c).$$

If we choose $0 < \epsilon_0 < 1/C(c)$ from the beginning, we have a contradiction.

Proposition 3.1 follows from Lemma 3.2 and Theorem 3 of [4].

Next proposition is concerning the decay of $u(x)$ at infinity.

Proposition 3.2: Suppose $u \in C^2(\mathbb{R}^2_+) \cap C^1(\bar{\mathbb{R}}^2_+)$ is a solution of (5) satisfying (7). Then

$$\lim_{|x|\to\infty} \frac{u(x)}{\log|x|} = d := -\frac{1}{\pi}\int_{\mathbb{R}^2_+} e^{u(x)}dx + \frac{c}{\pi}\int_{\partial\mathbb{R}^2_+} e^{u(x)/2}dx.$$

Proof: Let

$$\begin{aligned} w(x) = & \tfrac{1}{2\pi}\textstyle\int_{\mathbb{R}^2_+}(\log|x-y| + \log|\bar{x}-y| - 2\log|y|)e^{u(y)}dy \\ & -\tfrac{c}{2\pi}\textstyle\int_{\partial\mathbb{R}^2_+}(\log|x-y| + \log|\bar{x}-y| - 2\log|y|)e^{u(y)/2}dy. \end{aligned}$$

It is easy to check that w satisfies

$$\begin{cases} \Delta w = & e^u \quad \text{in } \mathbb{R}^2_+, \\ \frac{\partial w}{\partial t} = & -ce^{u/2} \quad \text{on } \partial\mathbb{R}^2_+, \end{cases}$$

and, using (7) and Proposition 3.1,

$$\lim_{|x|\to+\infty} \frac{w(x)}{\log|x|} = -d.$$

Considering $\bar{v}(x) = u(x) + w(x)$. Then

$$\begin{cases} \Delta \bar{v} = & 0 \quad \text{in } \mathbb{R}^2_+, \\ \frac{\partial \bar{v}}{\partial t} = & 0 \quad \text{on } \partial \mathbb{R}^2_+. \end{cases}$$

We extend $\bar{v}(x)$ to $\mathbb{R}^2$ by even reflection. From Proposition 3.1, we know $\bar{v}(x) \le C(1 + \log(|x| + 1))$ for some positive constant C. Thus $\bar{v}(x)$ is a constant. This completes the proof.

3.1 Case $c \ge 0$

Proposition 3.3 Suppose $u \in C^2(\mathbb{R}^2_+) \cap C^1(\bar{\mathbb{R}}^2_+)$ satisfies (5) and (7) with $c \ge 0$. Then $d = -4$, and for all $b \in \mathbb{R}$, there exists some $\lambda_b > 0$, such that

$$u_b(x) = u_b(\frac{\lambda_b^{-2} x}{|x|^2}) - 4\log(\frac{|x|}{\lambda_b^{-1}}), \qquad \forall\, x \in \mathbb{R}^2_+,$$

where $u_b(x) = u(s + b, t),\ \forall\, x = (s, t) \in \mathbb{R}^2_+$.

It is crucial to show that $d = -4$. We will show it by obtaining contradictions when assuming $d < -4$ or $d > -4$.

First we assume $d > -4$.

Set

$$v_b(x) = u_b(\frac{x}{|x|^2}) - 4\log|x|,$$

and

$$w_{\lambda,b}(x) = v_b(x) - (v_b(\frac{\lambda^2 x}{|x|^2}) - 4\log\frac{|x|}{\lambda}).$$

In the following we always write $v(x) = v_0(x)$, $v_\lambda(x) = v(\frac{\lambda^2 x}{|x|^2}) - 4\log\frac{|x|}{\lambda}$ and $w_\lambda(x) = v(x) - v_\lambda(x)$. So w_λ satisfies:

$$\begin{cases} \Delta w_\lambda + & c_1(x) w_\lambda = 0 \quad \text{in } B_\lambda^+, \\ \frac{\partial w_\lambda}{\partial t} = & c_2(x) w_\lambda \quad \text{on } \partial B_\lambda^+ \cap \partial \mathbb{R}^2_+, \end{cases} \tag{49}$$

where $c_1 = e^{\xi_1(x)}$, $c_2 = \frac{c}{2} e^{\frac{\xi_2}{2}}$, ξ_i $(i = 1, 2)$ are two functions between v_λ and v. We will derive a contradiction from the following four claims.

Claim 1: For λ large enough, $w_\lambda(x) \ge 0$ for all $x \in \overline{B}_\lambda^+ \setminus \{0\}$.

Proof: We prove this claim by three steps.

Step 1: $\exists R_0 > 0$, such that for all $R_0 \le |x| \le \lambda/2$, we have $w_\lambda(x) \ge 0$.

Proof: For $R_0 \le |x| \le \lambda/2$, R_0 large enough

$$\begin{aligned} w_\lambda(x) = & \; v(x) - (v(\tfrac{\lambda^2 x}{|x|^2}) - 4\log\tfrac{|x|}{\lambda}) \\ = & \; u(\tfrac{x}{|x|^2}) - 4\log|x| - (u(\tfrac{x}{\lambda^2}) - 4\log\tfrac{\lambda^2|x|}{|x|^2} - 4\log\tfrac{|x|}{\lambda}) \\ = & \; O(\tfrac{1}{|x|}) + 4\log\tfrac{\lambda}{|x|} \ge O(\tfrac{1}{|x|}) + 4\log 2 > 0. \end{aligned}$$

Step 2: $\exists R_1 \ge R_0$, such that for $R_1 \le \lambda/2 \le |x| \le \lambda$, we have $w_\lambda(x) \ge 0$.

Proof: Let $g(x) = \log(|x|-1)$, and $\bar{w}_\lambda(x) = w_\lambda(x)/g(x)$. From (49) we have

$$\begin{cases} \Delta\bar{w}_\lambda & +\frac{2}{g}\nabla g\cdot\nabla\bar{w}_\lambda + (c_1(x) + \frac{\Delta g}{g})\bar{w}_\lambda = 0 \quad in \quad B_\lambda^+, \\ \frac{\partial \bar{w}_\lambda}{\partial t} & = c_2(x)\bar{w}_\lambda \quad on \quad \{t=0\}\cap \overline{B}_\lambda^+ \setminus \{0\}. \end{cases}$$

Suppose the contrary, $\exists\, x_0 = (s_0, t_0)$ with $\lambda/2 \le |x_0| \le \lambda$, such that $\bar{w}_\lambda(x_0) = \min\limits_{\lambda/2\le|x|\le\lambda} \bar{w}_\lambda(x) < 0$. Then $|x_0| \ne \lambda$ from the definition of w_λ, $|x_0| \ne \lambda/2$ from Step 1. It follows that $\lambda/2 < |x_0| < \lambda$ and

$$\begin{aligned} v_\lambda(x_0) = & \; v(\tfrac{\lambda^2 x_0}{|x_0|^2}) - 4\log\tfrac{|x_0|}{\lambda} \\ = & \; u(\tfrac{x_0}{\lambda^2}) - 4\log\tfrac{\lambda^2}{|x_0|} - 4\log\tfrac{|x_0|}{\lambda} \\ \le & \; C - 4\log\lambda \le C_1 - 4\log|x_0|, \end{aligned}$$

$$v(x_0) \le v_\lambda(x_0) \le C_1 - 4\log|x_0|.$$

Thus, $c_1(x_0) \le \frac{C}{|x_0|^4}$. By direct calculation we have

$$\frac{\Delta g}{g}(x) = -\frac{1}{|x_0|(|x_0|-1)^2\log(|x_0|-1)}.$$

It is clear that for $R_1 \ge R_0$ large enough we have

$$c_1(x_0) + \frac{\Delta g}{g}(x_0) < 0.$$

Arguing as in Section 2.1, we have $t_0 > 0$, and reach a contradiction. Step 2 is established.

Step 3: $\exists\, R_2 \ge R_1$, such that for $\lambda \ge R_2$,

$$w_\lambda(x) \ge 0 \;\text{ for }\; x \in B_{R_0}^+ \setminus \{0\}.$$

Proof: For $0 \le |x| \le R_0$, $d > -4$, λ large enough we have

$$\begin{aligned} w_\lambda(x) = & \; v(x) - [v(\tfrac{\lambda^2 x}{|x|^2}) - 4\log\tfrac{|x|}{\lambda}] \\ = & \; u(\tfrac{x}{|x|^2}) - 4\log|x| - [u(\tfrac{x}{\lambda^2}) - 4\log\tfrac{\lambda^2}{|x|} - 4\log\tfrac{|x|}{\lambda}] \\ = & \; u(\tfrac{x}{|x|^2}) - 4\log|x| - u(\tfrac{x}{\lambda^2}) + 4\log\lambda. \end{aligned} \tag{50}$$

Notice that as $|x| \to 0$, $u(\frac{x}{|x|^2}) - 4\log|x| = (-d-4+o(1))\log|x| > 0$. So we know that

$$u(\frac{x}{|x|^2}) - 4\log|x| > -C \quad for \quad x \in B^+_{R_0} \setminus \{0\}.$$

Plugging it into (50), we have established Step 3. Claim 1 follows from Step 1-3.

Remark 3.1: We only used $d > -4$ in Step 3.

Now we define for $b \in \partial\mathbb{R}^2_+ = \mathbb{R}$ that

$$\lambda_b = \inf\{\lambda > 0 \mid w_{\mu,b}(x) \geq 0 \ \ in \ \bar{B}^+_\mu \setminus \{0\} \ \ for \ \ all \ \ \lambda < \mu < \infty\}.$$

Claim 2: $\exists \bar{b} \in \mathbb{R}$, such that $\lambda_{\bar{b}} > 0$.

Lemma 3.3: Suppose $f \in C^1(\mathbb{R}^2_+)$ satisfying, for all $b \in \mathbb{R}, \lambda > 0$,

$$f_b(x) - [f_b(\frac{\lambda^2 x}{|x|^2}) - 4\log\frac{|x|}{\lambda}] \leq 0, \ \forall \ x \in B^+_\lambda,$$

where $f_b(x) = f_b(s,t) = f(s+b,t)$, $\forall x = (s,t) \in \mathbb{R}^2_+$. Then

$$f(x) = f(s,t) = f(0,t), \qquad \forall \ x = (s,t) \in \mathbb{R}^2_+.$$

Proof: For all $b \in \mathbb{R}$, $\lambda > 0$, set

$$g_{b,\lambda}(x) \equiv f_b(x) - [f_b(\frac{\lambda^2 x}{|x|^2}) - 4\log\frac{|x|}{\lambda}], \ x \in \mathbb{R}^2_+, |x| \leq \lambda.$$

It is easy to see that for all $b \in \mathbb{R}$, $x \in \mathbb{R}^n_+$, we have

$$\begin{cases} g_{b,|x|}(x) & = 0, \\ g_{b,|x|}(rx) & \leq 0, \ \forall \ 0 < r < 1. \end{cases}$$

It follows that

$$\frac{d}{dr}\{g_{b,|x|}(rx)\}|_{r=1} \geq 0.$$

A direct computation yields

$$2\nabla f_b(x) \cdot x + 4 \geq 0,$$

Since b is arbitry, similarly as in lemma 2.2, we can deduce

$$\partial_s f(s,t) = 0, \ \forall s \in \mathbb{R}, \ t > 0.$$

We finish the proof.

Proof of Claim 2: If $\lambda_b = 0$ for all $b \in \mathbb{R}$, we have for all $b \in \mathbb{R}^{n-1}$ and $\lambda > 0$

$$v_b(x) \geq [v_b(\frac{\lambda^2 x}{|x|^2}) - 4\log\frac{|x|}{\lambda} \qquad \forall x \in B_\lambda^+.$$

It follows that for all $b \in \mathbb{R}^{n-1}$, and $\lambda > 0$ that

$$u_b(y) \leq [u_b(\frac{\lambda^{-2} y}{|y|^2}) - 4\log\lambda|x|] \qquad \forall y \in B_{1/\lambda}^+.$$

Therefore from Lemma 3.3 we know that $u(x)$ depends only on t. This obviously violates (7). We have verified Claim 2.

Claim 3: Suppose $\lambda_b > 0$ for some $b \in \partial\mathbb{R}_+^2 = \mathbb{R}$, then we have $w_{\lambda_b, b}(x) = 0, \ \forall\, x \in \mathbb{R}_+^2$.

Proof: Without loss of generality, we assume $b = 0$. Suppose the contrary of Claim 3, w_{λ_0} satisfies

$$\begin{cases} \Delta w_{\lambda_0} & \leq 0 & \text{in } B_{\lambda_0}^+, \\ \frac{\partial w_{\lambda_0}}{\partial t} & = ce^{\frac{v}{2}} - ce^{\frac{v_\lambda}{2}} & \text{on } \partial B_{\lambda_0}^+ \cap \partial\mathbb{R}_+^2 \setminus \{0\}, \\ w_{\lambda_0} & \geq 0, & \text{in } B_{\lambda_0}^+. \end{cases} \tag{51}$$

It follows from the strong maximum principle and the Hopf lemma (using also the boundary condition of w_{λ_0} given in (51)) that

$$\begin{cases} w_{\lambda_0}(x) > 0, & \forall\, x \in \bar{B}_{\lambda_0}^+, \ 0 < |x| < \lambda_0, \\ \frac{\partial w_{\lambda_0}}{\partial\nu}(x) > 0, & \forall\, x \in \partial B_{\lambda_0}^+ \cap \mathbb{R}_+^2, \end{cases} \tag{52}$$

where ν denotes the inner unit normal of the sphere ∂B_{λ_0}.

Lemma 3.4: There exists some constant $\gamma = \gamma(\lambda_0) > 0$ such that $w_{\lambda_0}(x) \geq \gamma$ for $x \in \overline{B}_{\lambda_0/2}^+ \setminus \{0\}$.

Proof: Notice that

$$\begin{aligned} v_{\lambda_0}(x) & = v(\tfrac{\lambda_0^2 x}{|x|^2}) - 4\log\tfrac{|x|}{\lambda_0} \\ & = u(\tfrac{x}{\lambda_0^2}) - 4\log\lambda_0 \leq C_1(\lambda_0). \end{aligned} \tag{53}$$

From (52) we have $\min\limits_{\partial B_{\frac{\lambda_0}{2}}^+ \cap \partial B_{\lambda_0/2}} w_{\lambda_0} \geq \epsilon$, for some $0 < \epsilon < 1$.Without loss of generality, we assume $\lambda_0 = 2$. For $0 < r < 1$, we introduce an auxilary function,

$$\varphi_4(x) = \frac{\epsilon\mu}{2(c+1)} - \frac{\log|x|}{\log r} \cdot \epsilon + \frac{\epsilon t(1-\mu)}{2}, \qquad x \in B_1^+ \setminus B_r^+,$$

where $0<\mu<1$ being chosen later. Let $P(x)=w_{\lambda_0}(x)-\varphi_4(x)$, then $P(x)$ satisfies

$$\begin{cases} \Delta P(x) & \leq 0 \qquad\qquad\qquad\qquad \text{in } B_1^+ \setminus \bar{B}_r^+, \\ \frac{\partial P(x)}{\partial t} & = ce^{\frac{v}{2}} - ce^{\frac{v_{\lambda_0}}{2}} - \frac{\epsilon(1-\mu)}{2} \quad \text{on } \partial(B_1^+ \setminus \bar{B}_r^+) \cap \partial\mathbb{R}_+^2. \end{cases}$$

We will show that

$$P(x) \geq 0, \qquad \forall\, x \in \bar{B}_1^+ \setminus B_r^+. \tag{54}$$

On $\partial B_1^+ \cap \partial B_1 : P(x) \geq \epsilon - \epsilon = 0$; on $\partial B_r^+ \cap \partial B_r : P(x) > w_{\lambda_0}(x) \geq 0$. If (54) does not hold, there exists some $x_0=(s_0,t_0)$ such that, $P(x_0)=\min\limits_{x\in\overline{B}_1^+\setminus B_r^+} P(x)<0$. It follows from the above consideration and the maximum principle that $t_0=0, r<|s_0|<1$. Then we have $\frac{\partial P(x)}{\partial t}(x_0)\geq 0$.

From $P(x_0)<0$, we have $v(x_0)-v_{\lambda_0}(x_0)-\varphi_4(x_0)<0$. It follows from (53) that

$$v(x_0) < C_2, \tag{55}$$

for some constant $C_2=C_2(\epsilon,C_1)$. Also

$$w_{\lambda_0}(x_0) < \frac{\epsilon\mu}{2(c+1)} - \frac{\log|x|}{\log r}\cdot\epsilon < \frac{\epsilon\mu}{2(c+1)}. \tag{56}$$

From (55),(53), and the mean value theorem we have $e^{\frac{v}{2}}-e^{\frac{v_{\lambda_0}}{2}}<C_3 w_{\lambda_0}(x_0)$for some constant $C_3>0$. Combining with $\frac{\partial P}{\partial t}(x_0)\geq 0$, we have

$$w_{\lambda_0}(x_0) \geq \frac{\epsilon}{2(c+1)C_3}\cdot(1-\mu). \tag{57}$$

Combining (56)and (57) we have

$$\mu > \frac{1}{1+C_3}.$$

If we choose μ such that $0<\mu<\frac{1}{1+C_3}$ from the beginning, We reach a contradiction. (54) is established. Let $r\to 0$, we have proved Lemma 3.4 with $\gamma=\frac{\epsilon}{2(1+c)(1+C_3)}$.

From the definition of λ_0, there exists a sequence $\lambda_k\to\lambda_0$ with $\lambda_k<\lambda_0$, such that

$$\inf_{\bar{B}_{\lambda_k}^+\setminus\{0\}} w_{\lambda_k} < 0.$$

It is not difficult to see from Lemma 3.2 and the continuity of u at 0 that for k large enough, we have

$$w_{\lambda_k}(x) \geq \gamma/2, \quad \forall\, x \in \overline{B}^+_{\lambda_0/2} \setminus \{0\}.$$

It follows that there exists $x_k = (s_k, t_k) \in \bar{B}^+_{\lambda_k} \setminus B^+_{\lambda_0/2}$ such that

$$w_{\lambda_k}(x_k) = \min_{\bar{B}^+_{\lambda_k} \setminus \{0\}} w_{\lambda_k} < 0.$$

It is clear that $\lambda_0/2 < |x_k| < \lambda_k$ and, due to the boundary condition, $t_k > 0$. Hence $\nabla w_{\lambda_k}(x_k) = 0$. After passing to a subsequence (still denoted as x_k) $x_k \to x_0 = (s_0, t_0)$. It follows that

$$w_{\lambda_0}(x_0) = 0, \qquad \nabla w_{\lambda_0}(x_0) = 0. \tag{58}$$

It follows from (58) and (52) that $t_0 = 0, |s_0| = \lambda_0$.

Lemma 3.5: If (51) and (52) hold. Then $\frac{\partial w_{\lambda_0}}{\partial \nu}(x_0) > 0$ for all $x_0 = (s_0, 0), |s_0| = \lambda_0$.

Once we establish Lemma 3.5, we will have reached a contradiction, thus have verified Claim 3.

Proof of Lemma 3.5: Without loss of generality we assume $\lambda_0 = 1, s_0 = 1$. Set $\Omega = \{x = (s,t) \mid 1/2 < s^2 + t^2 < 1, s > 0, 0 < t < 1/4\}$, and

$$h(x) = \epsilon(1-s)(t+\mu), \quad \varphi_5(x) = h(x) - h(\frac{x}{|x|^2}),$$

where $0 < \epsilon, \mu < 1$ being chosen later. A direct computation yields

$$\Delta \varphi_5(x) = 0, \quad \forall\, x \in \Omega.$$

Consider $B(x) = w_{\lambda_0}(x) - \varphi_5(x)$, it follows that

$$\begin{cases} \Delta B & \leq 0, & \text{in } \ \Omega, \\ \frac{\partial B}{\partial t} & = c_2(x) w_{\lambda_0}(x) - \frac{\partial \varphi_5}{\partial t} & \text{on } \partial\Omega \cap \partial\mathbb{R}^2_+, \end{cases} \tag{59}$$

where $c_2 = \frac{c}{2} e^{\frac{\xi_2}{2}}$, ξ_2 is a function between v_λ and v.

For suitably chosen ϵ and μ, we want to show

$$B(x) = w_{\lambda_0}(x) - \varphi_5(x) \geq 0, \quad \forall\, x \in \ \Omega. \tag{60}$$

Using (52), we can choose $\epsilon_0 > 0$ small enough, such that for all $0 < \epsilon < \epsilon_0$, we have $B(x) \geq 0$ on $\partial\Omega \cap \{\partial B_{1/2} \cup \{t = 1/4\}\}$. Also from the construction of φ_5, we know $B(x) = 0$ on $\Omega \cap \partial B_1$. Suppose the contrary of (60), there exists some $x_1 = (s_1, t_1) \in \bar{\Omega}$ such that

$$B(x_1) = \min_{\bar{\Omega}} B < 0. \tag{61}$$

From the above and the maximum principle, we have $t_1 = 0, 1/2 < s_1 < 1$. Thus

$$\frac{\partial B}{\partial t}(x_1) \geq 0. \tag{62}$$

A simple calculation yields

$$\frac{\partial \varphi_5}{\partial t}(x_1) = \epsilon(1 - s_1)(s_1^{-3} + 1). \tag{63}$$

Combining (59), (62) and (63) we have

$$c_2(x_1) w_{\lambda_0}(x_1) - \epsilon(1 - s_1)(s_1^{-3} + 1) \geq 0. \tag{64}$$

It follows from (61) that

$$w_{\lambda_0}(x_1) < \epsilon\mu(1 - s_1)(s_1^{-1} + 1). \tag{65}$$

Combining (64) and (65) we have

$$c_2(x_1)\mu > 1.$$

So if we choose $0 < \mu < \min_{1/2 \leq |x| \leq 1} (c_2(x) + 1)^{-1}$ from the beginning, we reach a contradiction. Thus (60) holds. Since we also know that $B(x_0) = 0$, we have

$$\frac{\partial B}{\partial \nu}(x_0) \geq 0.$$

It follows from a direct computation that

$$\frac{\partial w_{\lambda_0}}{\partial \nu}(x_0) = \frac{\partial B}{\partial \nu}(x_0) + \frac{\partial \varphi_5}{\partial \nu}(x_0) \geq \frac{\partial \varphi_5}{\partial \nu}(x_0) = 2\epsilon\mu > 0$$

Lemma 3.2 is established, also Claim 3 as remarked earlier.

From Claim 2 and Claim 3, it is easy to see that $d = -4$. This contradicts to our assumption $d > -4$. We have thus proved $d \leq -4$.

Lemma 3.6: For $d \le -4$, $\alpha := \lim_{|x|\to\infty}[u(x) - d\log|x|]$ exists and is finte. Furthermore

$$|u(x) - d\log|x| - \alpha| \le \frac{C\log|x|}{|x|}, \quad \forall\, x \in \mathbb{R}^2_+,\ |x| \ge 1.$$

Proof: Since $d \le -4$, it's easy to see that for $|y| \ge 1$, $e^{u(y)} \le C|y|^{-4}$, $e^{u(y)/2} \le C|y|^{-2}$. From the proof of Proposition 3.2 and the above decay property of e^u, we have for some constant α

$$\begin{aligned} u(x) - d\log|x| - \alpha = \quad & -\tfrac{1}{2\pi}\textstyle\int_{\mathbb{R}^2_+}(\log|x-y| + \log|\bar{x}-y| - 2\log|x|)e^{u(y)}dy \\ & +\tfrac{c}{2\pi}\textstyle\int_{\partial\mathbb{R}^2_+}(\log|x-y| + \log|\bar{x}-y| - 2\log|x|)e^{u(y)/2}dy. \end{aligned}$$

Lemma 3.6 follows from some elementary calculations.

With Lemma 3.6, we simply repeat the previous moving sphere argument for u instead of for v and we will also reach a contradiction if $d < -4$. One difference we need to point out is in Step 3. For this step, $d < -4$ for u will play the same role as $d > -4$ for v. The other difference is in Step 1. We leave the details to readers.

Now we have proved that $d = -4$. In this case, we still work with the Kelvin transformation v of u and can see that Claim 1 to Claim 3 are still valid. The reason is that, as pointed out in Remark 3.1, we only need to establish Step 3 when $d = -4$. But this can be done without much difficulty with the help of Lemma 3.6. We leave the details to readers.

Claim 4: For all $b \in \mathbb{R}$, we have $\lambda_b > 0$.

Proof: It follows from Claim 2 and Claim 3 that there exists some $\bar{b} \in \mathbb{R}$ such that $\lambda_{\bar{b}} > 0$ and $w_{\lambda_{\bar{b}},\bar{b}}(x) = 0,\ \forall\, x \in \mathbb{R}^2_+$. That is

$$u_{\bar{b}}(x) = u_{\bar{b}}\left(\frac{x}{\lambda^2_{\bar{b}}|x|^2}\right) - 4\log(\lambda_{\bar{b}}|x|)\ \ \forall\, x \in \mathbb{R}^2_+.$$

It follows that

$$\lim_{|x|\to\infty}[u_{\bar{b}}(x) + 4\log|x|] = u(\bar{b}, 0) - 4\log\lambda_{\bar{b}}.$$

namely,

$$\lim_{|x|\to\infty}[u(x) + 4\log|x|] = u(\bar{b}, 0) - 4\log\lambda_{\bar{b}}. \tag{66}$$

Suppose the contrary of Claim 4 for some $b \in \mathbb{R}$, namely,

$$w_{\lambda,b}(x) = v_b(x) - \left[v_b\left(\frac{\lambda^2 x}{|x|^2}\right) - 4\log\frac{|x|}{\lambda}\right] \ge 0, \quad \forall\, \lambda > 0,\ \forall x \in \bar{B}^+_\lambda \setminus \{0\}.$$

It follows that

$$u_b(\frac{x}{|x|^2}) \geq u_b(\frac{x}{\lambda^2}) - 4\log\frac{\lambda}{|x|}, \quad \forall\ \lambda > 0,\ \forall x \in \bar{B}_\lambda^+ \setminus \{0\}.$$

Fixing $\lambda > 0$ in the above and sending $|x|$ to 0, we have (using (66))

$$u(\bar{b}, 0) - 4\log\lambda_{\bar{b}} \geq u_b(0) - 4\log\lambda.$$

Sending λ to 0, we get a contradiction. Claim 4 has been verified.

Proposition 3.3 follows from Claim 1 to Claim 4.

Lemma 3.7: Suppose $f \in C^1(\mathbb{R})$ satisfies: $\forall\ b \in \mathbb{R}$, there exists some $\mu_b > 0$ such that

$$f(s+b) = f(\frac{\mu_b^2}{s} + b) - 4\log(\frac{|s|}{\mu_b}), \quad \forall\ s \in \mathbb{R} \setminus \{0\}.$$

Then for some $s_0 \in \mathbb{R}, a, d > 0$,

$$f(s) = \log(\frac{a}{(s-s_0)^2 + d})^2, \quad \forall\ s \in \mathbb{R}.$$

Proof: Considering

$$h(s) = e^{-f(s)/2}, \quad s \in \mathbb{R}.$$

For all $b \in \mathbb{R}$, we have

$$h(s+b) = \frac{s^2}{\mu_b^2} h(\frac{\mu_b^2}{s} + b), \quad \forall\ s \in \mathbb{R} \setminus \{0\}.$$

For $|s|$ large, we have

$$h(s) = \frac{s^2}{\mu_0^2}\{h(0) + h'(0)\frac{\mu_0^2}{s} + O(\frac{1}{s^2})\},$$

$$h(s) = \frac{(s-b)^2}{\mu_b^2}\{h(b) + h'(b)\frac{\mu_b^2}{s-b} + O(\frac{1}{s^2})\}.$$

Comparing the above two formula, we have

$$\frac{h(b)}{\mu_b^2} = \frac{h(0)}{\mu_0^2}, \quad h'(b) - \frac{2bh(b)}{\mu_b^2} = h'(0).$$

It follows that

$$h(b) = \frac{h(0)}{\mu_0^2} b^2 + h'(0)b + h(0), \quad \forall\, b \in \mathbb{R}.$$

Therefore for some $s_0, d \in \mathbb{R}, a > 0$, we have

$$h(s) = \frac{1}{a}\{(s - s_0)^2 + d\}.$$

Clearly $d > 0$ and Lemma 3.7 follows.

Proof of Theorem 1.3 when $c \geq 0$: Using Proposition 3.3 and Lemma 3.7, we know that for some $s_0 \in \mathbb{R}, a, d > 0$ we have

$$u(s, 0) = \log(\frac{a}{(s - s_0)^2 + d})^2, \ \forall\, s \in \mathbb{R}.$$

Set $x_0 = (s_0, -\sqrt{d})$, $B = \{(s,t) : \frac{t+\sqrt{d}}{|s-s_0|^2+(t+\sqrt{d})^2} - \sqrt{d} \geq 0\}$, and

$$\varphi(x) = u(\frac{x - x_0}{|x - x_0|^2} + x_0) - 4\log|x - x_0|, \quad x \in B.$$

It is clear that B is a ball in $\mathbb{R}^2$. On the boundry of that ball $\partial B = \{(s,t) : \frac{t+\sqrt{d}}{|s-s_0|^2+(t+\sqrt{d})^2} - \sqrt{d} = 0\}$, we have

$$\begin{aligned}\varphi(s,t) = \ & 2\log a - 2\log[\tfrac{|s-s_0|^2}{(|s-s_0|^2+(t+\sqrt{d})^2)^2} + d] - 2\log[|s-s_0|^2 + (t+\sqrt{d})^2] \\ & = 2\log a - 2\log\{\tfrac{|s-s_0|^2}{|s-s_0|^2+(t+\sqrt{d})^2} + d \cdot [|s-s_0|^2 + (t+\sqrt{d})^2]\} \\ & = 2\log a.\end{aligned}$$

Using the maximum principle and the result of Gidas, Ni and Nirenberg [15], we conclude that $w - 2\log a \geq 0$ in B and w is radially symmetric about the center of B. Theorem 1.3 in this case follows basically the same way as in the proof of Theorem 1.1.

3.2 Case $c < 0$.

Proof of Theorem 1.3 in Case $c < 0$: From Section 2.2 and section 3.1, we can see that almost all steps are the same as in Section 3.1. The main difference between Case $c \geq 0$ and $c < 0$ when $n = 2$ is to show Step 2 in the proof of Claim 1. Actually we can prove this step in Case $c < 0$ by using test fuction $g(x) = \log(|z|-1)$ instead of $g(x) = \log(|x|-1)$ with $z = x + (0, -\lambda/4)$. The main procedure of this case is similar to Case $c > 0$, so we define $v(x)$,$v_b(x)$,$w_{\lambda,b}$ and w_λ as in Section 3.1.

Proposition 3.5 If $u(x)$ satisfies (5) and (7), then $d = -4$ and for all $b \in \mathbb{R}$, there exists some $\lambda_b > 0$, such that

$$u_b(x) = u_b(\frac{\lambda_b^{-2} x}{|x|^2}) - 4\log(\frac{|x|}{\lambda_b^{-1}}), \qquad \forall\, x \in \mathbb{R}^2_+,$$

where $u_b(x) = u(s+b, t)$.

Similarly we will deduce contradiction if $d < -4$ and $d > -4$.

First we assume $d > -4$. We will derive a contradiction as in Section 3.1.

Claim 1: If λ is large enough, then $w_\lambda(x) \geq 0$ for all $x \in \overline{B}_\lambda^+ \setminus \{0\}$.

Proof: We still prove this claim by three steps.

Step 1: $\exists R_0 > 0$, such that for all $R_0 \leq |x| \leq \lambda/2$, we have $w_\lambda(x) \geq 0$.

Proof: See Section 3.1 (Step 1 in Claim 1).

Step 2: $\exists R_1 \geq R_0$, such that for $R_1 \leq \lambda/2 \leq |x| \leq \lambda$, we have $w_\lambda(x) \geq 0$.

Proof: Let $g(x) = \log(|z|-1)$ with $z = x + (0, -\frac{\lambda}{4})$ and $\bar{w}_\lambda(x) = w_\lambda(x)/g(x)$. From (49) we have

$$\begin{cases} \Delta \bar{w}_\lambda & +\frac{2}{g}\nabla g \cdot \nabla \bar{w}_\lambda + (c_1(x) + \frac{\Delta g}{g})\bar{w}_\lambda = 0 \quad in \;\; B_\lambda^+, \\ \frac{\partial \bar{w}_\lambda}{\partial t} & = (c_2(x) - \frac{1}{g}\cdot\frac{\partial g}{\partial t})\bar{w}_\lambda \quad on \;\; \{t=0\} \cap \overline{B}_\lambda^+ \setminus \{0\}. \end{cases} \tag{67}$$

Suppose the contrary, $\exists\ x_0 = (s_0, t_0)$ with $\lambda/2 \leq |x_0| \leq \lambda$, such that $\bar{w}_\lambda(x_0) = \min\limits_{\lambda/2 \leq |x| \leq \lambda} \bar{w}_\lambda(x) < 0$. Then $|x_0| \neq \lambda$ from the definition of w_λ, $|x_0| \neq \lambda/2$ from Step 1, and $t_0 > 0$ due to the boundary condition. It follows that $x_0 \in B_\lambda^+ \setminus \bar{B}_{\lambda/2}^+$ and

$$v_\lambda(x_0) \leq C - 4\log\lambda \leq C_1 - 4\log|x_0|,$$

$$v(x_0) \leq v_\lambda(x_0) \leq C_1 - 4\log|x_0|.$$

Thus, when λ is large enough

$$|c_1(x_0)| \leq \frac{C_1}{|x_0|^4}, \quad |c_2(x)| \leq \frac{C_2}{|x_0|^2}, \quad |x_0| \sim |z_0| \sim \lambda,$$

where $z_0 = x_0 + (0, -\lambda/4)$.

By a direct calculation we have

$$\frac{\Delta g}{g}(x_0) = -\frac{1}{|z_0|(|z_0|-1)^2 \log(|z_0|-1)},$$

$$\frac{1}{g} \cdot \frac{\partial g}{\partial t}(x_0) = -\frac{\lambda}{4|z_0|(|z_0|-1)\log(|z_0|-1)}.$$

Then as λ large enough, we have

$$c_1(x_0) + \frac{\Delta g}{g}(x_0) < 0,$$

and

$$c_2(x) - \frac{1}{g} \cdot \frac{\partial g}{\partial t}(x_0) > 0.$$

Step 2 can be established as before.

Step 3: $\exists\ R_2 \geq R_1$, such that for $\lambda \geq R_2$,

$$w_\lambda(x) \geq 0 \quad \text{for} \quad x \in B_{R_0}^{+} \setminus \{0\}.$$

Proof: See Section 3.1 (Step 3 in Claim 1).

The rest of the proof is very similar to that in Section 3.1, changing test functions to some standard comparision functions in the proof of Claim 3 as in Section 2.2. Then we can prove Proposition 3.5.

Theorem 1.3 in Case $c < 0$ follows by some simple modifications of previous argument.

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