Chapter 17 Complex numbers

Introduction

- Complex numbers (Definition 17.2)
- Modulus and argument angle
- Principle n-th root

17.1 Definition of complex numbers

Back to radical expressions, what if the base a in the expression a^p is negative? Is it useful? If yes, then how to understand this number? We start from one of the simplest forms $(-1)^{1/2}$ or $\sqrt{-1}$.

Definition 17.1. Complex symbol i

The complex symbol i is defined as

$$i = \sqrt{-1}.$$

♣

The main property for i is of course that $i^2 = -1$, or i is one solution to equation

$$x^2 = -1. \tag{17.1}$$

A general complex number is introduced in the following

Definition 17.2. Complex number

For given real numbers a, b, number $a + bi$ is called a complex number. ♣

The introduction of complex numbers eventually makes the number system completed in a certain mathematical sense (kind of mysterious, right? Do not feel bad, after all, not everyone needs to be a mathematician). In particular, it allows mathematicians to establish the **Fundamental Theorem of Algebra**: Every polynomial equation with complex coefficients and degree ≥ 1 has at least one complex root. The above theorem is used to be proved in an advanced college-level mathematical course called "Complex analysis". We sure omit proof here. So quite curious, one may ask: what is another solution to equation (17.1)?

17.2 Modulus and Argument

To answer the above question, we will describe another way to express a complex number (pre-calculus knowledge is needed). Let $z = a + bi$, where a, b are two real numbers. Then

$$z = a + bi = \sqrt{a^2 + b^2} \cdot \left(\frac{a}{\sqrt{a^2 + b^2}} + \frac{bi}{\sqrt{a^2 + b^2}}\right).$$

$\sqrt{a^2+b^2}$ is called the modulus of complex number z, denoted by $|z|$. If we write $\cos\theta = \frac{a}{\sqrt{a^2+b^2}}$, and $\sin\theta = \frac{a}{\sqrt{a^2+b^2}}$ for an angle θ in the range from zero to 2π, this angle θ is called the argument of complex number z. Then, mathematicians can prove that

$$z = |z|e^{\theta i}.$$

The argument of a complex number is not uniqueness (since $e^{2\pi i} = 1$). We thus call the argument of a complex number the "principal argument" if it is greater than or equal to 0 and is less than 2π.

17.3 All solutions to an algebraic equation

It is not too hard to see that

$$-1 = \cos\pi + i\sin\pi = e^{\pi i}.$$

So

$$x_1 = e^{\frac{\pi i}{2}} = \cos\frac{\pi}{2} + \sin\frac{\pi i}{2} = i$$

is one solution to equation (17.1) (here we use the principal argument of -1, we thus call this solution the principal square root of -1), as well as

$$x_2 = e^{\frac{3\pi i}{2}} = \cos\frac{3\pi}{2} + \sin\frac{3\pi i}{2} = -i$$

is the second solution.

The final exercise is the following

Example 17.1 Find all four solutions to

(1).

$$x^4 = 1;$$

and

(2).

$$x^4 = -1.$$

Solution: (1). Note

$$1 = e^{2\pi k i}, \quad \text{for } \; k = 1, 2, 3, 4.$$

So

$$x_k = e^{\frac{2\pi k}{4} i}, \quad \text{for } \; k = 1, 2, 3, 4,$$

all satisfies $x^4 = 1$. That is:

$$x_1 = e^{\frac{2\pi}{4} i} = i, \text{(the principal 4-th root of 1)}$$

$$x_2 = e^{\frac{4\pi}{4} i} = -1,$$

$$x_3 = e^{\frac{6\pi}{4} i} = -i,$$

$$x_2 = e^{\frac{8\pi}{4}i} = 1.$$

(2).

$$-1 = e^{2\pi ki+\pi i}, \quad \text{for} \;\; k = 1, 2, 3, 4.$$

So

$$x_k = e^{\frac{2\pi k}{4}i+\frac{\pi}{4}i}, \quad \text{for} \;\; k = 1, 2, 3, 4,$$

all satisfies $x^4 = -1$. That is:

$$x_1 = e^{\frac{2\pi}{4}i+\frac{\pi}{4}i} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i, \text{(the principal 4-th root of -1)}$$

$$x_2 = e^{\frac{4\pi}{4}i+\frac{\pi}{4}i} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i,$$

$$x_3 = e^{\frac{6\pi}{4}i+\frac{\pi}{4}i} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i,$$

$$x_4 = e^{\frac{8\pi}{4}i+\frac{\pi}{4}i} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i.$$

□

My note **Date:**

Chapter 17 Exercises

1. Basic skills: Find the modulus and phase angle.

   (a). $-i$

   (b). $1 - i$

(c). $\frac{\sqrt{3}}{2} + \frac{1}{2}i$

2. Basic skills: Solve equations (find all complex solutions)

(a). $x^3 = -1$

(b). $x^5 = 32$

(c). $x^2 = \frac{1}{2} + \frac{\sqrt{3}}{2}i$