SHARP HARDY-LITTLEWOOD-SOBOLEV INEQUALITY ON THE UPPER HALF SPACE

JINGBO DOU AND MEIJUN ZHU

Abstract There are at least two directions concerning the extension of classical sharp Hardy-Littlewood-Sobolev inequality: (1) Extending the sharp inequality on general manifolds; (2) Extending it for the negative exponent $\lambda = n - \alpha$ (that is for the case of $\alpha > n$). In this paper we confirm the possibility for the extension along the first direction by establishing the sharp Hardy-Littlewood-Sobolev inequality on the upper half space (which is conformally equivalent to a ball). The existences of extremal functions are obtained; And for certain range of the exponent, we classify all extremal functions via the method of moving sphere.

Keywords Hardy-Littlewood-Sobolev inequality; Conformal Laplacian operator; Green's function; Sharp constant; Moving sphere method
Mathematics Subject Classification(2010). 35A23, 42B37

1. Introduction

The classical sharp Hardy-Littlewood-Sobolev (HLS) inequality ([29, 30, 44, 39]) states that

$$|\int\int f(x)|x-y|^{-(n-\alpha)}g(y)dxdy| \le N(p,\alpha,n)||f||_p||g||_t \tag{1.1}$$

for all $f \in L^p(\mathbb{R}^n)$, $g \in L^t(\mathbb{R}^n)$, $1 < p,\ t < \infty$, $0 < \alpha < n$ and $1/p+1/t+(n-\alpha)/n = 2$. Lieb [39] proved the existence of the extremal function to the inequality with sharp constant, and computed the best constant in the case of $t = p$. The sharp HLS inequality implies Moser-Trudinger-Onofri and Beckner inequalities [2], as well as Gross's logarithmic Sobolev inequality [24]. All these inequalities play significant roles in solving global geometric problems, such as Yamabe problem, Ricci flow problem, etc. Besides the recent extension of the sharp HLS on the Heisenberg group by Frank and Lieb [21], there are at least two directions concerning the extension of the above sharp HLS inequality: (1) Extending the sharp inequality on general manifolds; (2) Extending it for the negative exponent $\lambda = n-\alpha$ (that is for the case of $\alpha > n$). In this paper, we study the extension of sharp HLS to the upper half space $\mathbb{R}^n$.

We start with manifolds with boundary. One of the simplest manifolds with boundary is the upper half space, or under the conformal equivalence, a ball in $\mathbb{R}^n$. By introducing an extension operator, we establish the sharp HLS inequality on the upper half space, and prove the existences of extremal functions; For certain exponent, we classify all extremal functions via the method of moving sphere, which was introduced in early work of Li and Zhu [36]. The current work builds a solid foundation for extending the classical sharp HLS on general manifolds. We shall outline (without proof) the general approach for such extensions later in this introduction.

1.1. **Singular integral operator on the upper half space.** Let $f(x) \in C_0^2(\mathbb{R}^n)$ for $n \geq 3$. The pointwise defined potential equation

$$-\Delta u = f$$

is equivalent to, up to a harmonic function, the following (globally defined) integral equation:

$$u(x) = \frac{1}{n(n-2)\omega_n} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-2}} dy,$$

where, and throughout the paper, ω_n denotes the volume of the $n-$dimensional unit ball. Generally, for all $\alpha > 0$, a positive solution $u(x) \in H^{\alpha/2}(\mathbb{R}^n)$ to

$$(-\Delta)^{\frac{\alpha}{2}} u = f, \quad u \in H^{\alpha/2}(\mathbb{R}^n) \tag{1.2}$$

in the distribution sense is given by

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{4}} u(x) (-\Delta)^{\frac{\alpha}{4}} \phi(x) dx = \int_{\mathbb{R}^n} \phi(x) f(x) dx,$$

for all $\phi(x) \in C_0^\infty(\mathbb{R}^n)$, where

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{4}} u(x) (-\Delta)^{\frac{\alpha}{4}} \phi(x) dx = \int_{\mathbb{R}^n} |\xi|^\alpha \widehat{u}(\xi) \overline{\widehat{\phi}}(\xi) d\xi.$$

If α is not an even number, equation (1.2) is globally defined. It is also known that it is equivalent to the integral equation (see, e.g., Stein [45] on P_{117})

$$u(x) = \frac{1}{c(n,\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \tag{1.3}$$

where $c(n,\alpha) = \pi^{\frac{n}{2}} 2^\alpha \Gamma(\frac{\alpha}{2}) / \Gamma(\frac{n-\alpha}{2})$. See other related work in Chen and Li [11].

For $f = u^{\frac{n+\alpha}{n-\alpha}}$, equation (1.3) is also satisfied by the extremal functions to certain sharp HLS inequality for the singular integral operator

$$I_\alpha f(x) = \frac{1}{c(n,\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

see, e.g., [39, 12, 34].

Parallel to the potential equation in the whole space, we consider the Laplacian equation on the upper half space with Neumann boundary condition. Let $f(y) \in C_0^2(\mathbb{R}^{n-1})$ for $n \geq 3$. The pointwise defined partial differential equation

$$\begin{cases} -\Delta u(x', x_n) = 0, & \text{for } \ x_n > 0 \text{ and } x' \in \mathbb{R}^{n-1}, \\ u_{x_n}(x', 0) = -f(x'), & \text{for } \ \ x' \in \mathbb{R}^{n-1} \end{cases}$$

is equivalent to, up to a harmonic function and a constant multiplier, the following integral equation:

$$u(x) = \int_{\partial \mathbb{R}^n_+} \frac{f(y)}{|x-y|^{n-2}} dy, \ \forall x = (x', x_n) \in \mathbb{R}^n_+, \tag{1.4}$$

where and throughout the paper, $|x-y| = \sqrt{|x'-y|^2 + x_n^2}$ for $x = (x', x_n) \in \mathbb{R}^n_+$ and $y \in \partial \mathbb{R}^n_+ = \mathbb{R}^{n-1}$. Equation (1.4) can also be viewed as another type of harmonic extensions of $f(y)$.

Generally, for $\alpha \in (1, n)$, we can introduce an extension operator for $f(y) \in C_0^\infty(\mathbb{R}^{n-1})$ as

$$E_\alpha f(x) = \int_{\partial \mathbb{R}^n_+} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \ \forall x = (x', x_n) \in \mathbb{R}^n_+. \tag{1.5}$$

It will be clear that the above extension is in fact a Laplacian type extension operator for even number α (see Proposition 5.5 in Section 5). Other properties about this extension and its relation to fractional Laplacian operator will be discussed in Section 5.

In the meantime, we consider the dual operator R_α for $g(x) \in C^\infty(\mathbb{R}^n_+)$ with compact set in $B_R(0)$ for large R:

$$R_\alpha g(y) = \int_{\mathbb{R}^n_+} \frac{g(x)}{|x-y|^{n-\alpha}} dx, \ \forall x = (x', x_n) \in \mathbb{R}^n_+, \ y \in \partial \mathbb{R}^n_+. \tag{1.6}$$

R_α can be viewed as a restriction operator.

The main goal of this paper is to establish the sharp HLS type inequalities for operators E_α and R_α.

Theorem 1.1. *For any* $1 < \alpha < n, 1 < p < \frac{n-1}{\alpha-1}$, *and*

$$\frac{1}{q} = \frac{n-1}{n}\left(\frac{1}{p} - \frac{\alpha-1}{n-1}\right), \tag{1.7}$$

there is a best constant $C_e(n, \alpha, p) > 0$ *depending on* n, α *and* p, *such that*

$$\|E_\alpha f\|_{L^q(\mathbb{R}^n_+)} \le C_e(n, \alpha, p) \|f\|_{L^p(\partial \mathbb{R}^n_+)}, \tag{1.8}$$

and the equality holds for certain extremal functions. Moreover, all extremal functions are radially symmetric (with respect to some points).

Note that inequality (1.8) is equivalent to the following HLS inequality.

Theorem 1.2. *(***HLS inequality on the upper half space***). For any* $1 < \alpha < n$, *and* $1 < t, \, p < \infty$ *satisfying*

$$\frac{n-1}{n} \cdot \frac{1}{p} + \frac{1}{t} + \frac{n-\alpha+1}{n} = 2, \tag{1.9}$$

the following sharp inequality holds for all $f \in L^p(\partial \mathbb{R}^n_+)$, $g \in L^t(\mathbb{R}^n_+)$:

$$\int_{\mathbb{R}^n_+} \int_{\partial \mathbb{R}^n_+} \frac{g(x) f(y)}{|x-y|^{n-\alpha}} dy dx \le C_e(n, \alpha, p) \|f\|_{L^p(\partial \mathbb{R}^n_+)} \|g\|_{L^t(\mathbb{R}^n_+)}. \tag{1.10}$$

This yields the sharp inequality for R_α.

Corollary 1.3. *For* $1 < \alpha < n$, $1 < t < \frac{n}{\alpha}$, *and*

$$\frac{1}{q} = \frac{n}{n-1}\left(\frac{1}{t} - \frac{\alpha}{n}\right), \tag{1.11}$$

there is a best constant $C_r(n, \alpha, t) > 0$ *depending on* n, α *and* t, *such that*

$$\|R_\alpha g\|_{L^q(\partial \mathbb{R}^n_+)} \le C_r(n, \alpha, t) \|g\|_{L^t(\mathbb{R}^n_+)}, \tag{1.12}$$

and the equality holds for certain extremal functions. Moreover, $C_r(n, \alpha, t) = C_e(n, \alpha, p)$, *where* p *is given via* (1.9).

The best constants in (1.8) can be classified as

$$C_e(n,\alpha,p) = \sup\{\|E_\alpha f\|_{L^q(\mathbb{R}^n_+)} \ : \ \|f\|_{L^p(\partial\mathbb{R}^n_+)} = 1\}.$$

The extremal functions to inequality (1.8), up to a positive constant multiplier, satisfy the following integral equation:

$$f^{p-1}(y) = \int_{\mathbb{R}^n_+} \frac{(E_\alpha f(x))^{q-1}}{|x-y|^{n-\alpha}} dx, \qquad \forall\, y \in \partial\mathbb{R}^n_+. \tag{1.13}$$

Using the method of moving sphere, we are able to classify all positive solutions to the above equation for certain power p, and obtain the precise value for the best constant.

Theorem 1.4. *For $1 < \alpha < n$, $p = \frac{2(n-1)}{n+\alpha-2}$, and $q = \frac{2n}{n-\alpha}$, if $f \in L_{loc}^{\frac{2(n-1)}{n+\alpha-2}}(\partial\mathbb{R}^n_+)$ is a positive solution to equation (1.13), then $f \in C^\infty(\partial\mathbb{R}^n_+)$, and must be the form of*

$$f(y) = c(n,\alpha)\Big(\frac{1}{|y-y_0|^2 + d^2}\Big)^{\frac{n+\alpha-2}{2}}, \qquad \forall\, y \in \partial\mathbb{R}^n_+$$

for some constants $c(n,\alpha)$, $d > 0$, and $y_0 \in \partial\mathbb{R}^n_+$. Thus, $C_e(n,\alpha,p)$ can be computed explicitly, in particular, for $\alpha = 2$,

$$C_e(n, 2, \frac{2(n-1)}{n}) = n^{\frac{n-2}{2(n-1)}} \omega_n^{1-\frac{1}{n}-\frac{1}{2(n-1)}}.$$

We do not know whether similar inequalities hold for $\alpha = 1$ or not. On the other hand, a limiting inequality for $\alpha = n$ can be obtained. See more remark late in this introduction and Section 5.2.

It will be clear in classifying all positive solutions to (1.13) via the method of moving sphere, that for $p = \frac{2(n-1)}{n+\alpha-2}$ and $q = \frac{2n}{n-\alpha}$, inequality (1.8) is equivalent to an integral inequality on a ball $B_r(0)$:

$$\|\tilde{E}_\alpha f\|_{L^q(B_r(0))} \le C_e(n,\alpha,p)\|f\|_{L^p(\partial B_r(0))}, \tag{1.14}$$

where

$$\tilde{E}_\alpha f(x) = \int_{\partial B_r(0)} \frac{f(y)}{|x-y|^{n-\alpha}} dS_y, \ \forall\, x \in B_r(0),$$

and one extremal function is always constant on the boundary. Moreover, for $\alpha = 2$, we can show that $\tilde{E}_\alpha f(x)$ is a constant function if extremal function $f(x)$ is a constant on the boundary. These enable us to obtain the best constant $C_e(n,\alpha,p)$ from inequality (1.14).

As a byproduct to our study, we also establish the following nonexistence result to (1.13) with subcritical exponents.

Theorem 1.5. *Let $1 < \alpha < n$, $\frac{2(n-1)}{n+\alpha-2} \le p < 2$, and $2 < q \le \frac{2n}{n-\alpha}$ satisfying*

$$\frac{1}{p} - \frac{n}{q(n-1)} < \frac{\alpha-1}{n-1}.$$

If $f \in L_{loc}^{\frac{(n-1)(2-p)}{\alpha-1}}(\partial\mathbb{R}^n_+)$ is a nonnegative solution to equation (1.13), then $f(x) = 0$.

1.2. **Sharp HLS on general manifolds.** Our current work has a strong implication in the extension of sharp HLS on general manifolds.

For a smooth and bounded domain $\Omega \subset \mathbb{R}^n$, we certainly can introduce

$$\tilde{E}_\alpha f(x) = \int_{\partial\Omega} \frac{f(y)}{|x-y|^{n-\alpha}} dS_y, \ \forall\, x \in \Omega$$

for smooth functions $f(y)$ defined on the boundary of Ω. Using the tools built in this paper, one shall be able to show that

$$\|\tilde{E}_\alpha f\|_{L^q(\Omega)} \le C_\Omega(n, \alpha, p)\|f\|_{L^p(\partial\Omega)}, \tag{1.15}$$

with the sharp constant $C_\Omega(n, \alpha, p)$ that is related to the geometric property of the domain. Using certain standard bubbling sequence of functions, we can show that

$$C_e(n, \alpha, p) \le C_\Omega(n, \alpha, p).$$

One may ask: for which type of domains will the strict inequality hold? Note that similar extension operators were studied by Hang, Wang and Yan [27], where they used the classical Riesz potential (corresponding to partial differential equations with Dirichlet boundary condition); And similar question was discussed by them (Conjecture 1.1 in [28]).

However, the danger for linking $C_\Omega(n, \alpha, p)$ to the the geometric property of the domain is that the geometric property of a general domain does not play essential role in the definition of the extension operator $\tilde{E}_\alpha$. Similar question can be asked for other kernels on manifolds (for example, Stein and Weiss type kernel [46]). Broadly, we may ask, how to extend the sharp HLS on general manifolds.

For $\alpha = 2$, we observe: the common property for the kernel to our inequality (1.10) and that to the classical HLS (1.1) is that both of them, up to a constant multiplier, are the Green's function of corresponding conformal Laplacian operators.

We thus propose (for simplicity we only consider $\alpha = 2$ here) to extend the classical Sharp HLS inequality as follows. Let (M^n, g) (for $n \ge 3$) be a compact smooth Riemmanian manifold, and R_g be its scalar curvature. Let $G_y(x)$ be the Green's function to the conformal Lapacian operator $L_g := -\Delta_g + \frac{n-2}{4(n-1)} R$ with pole at $y \in M^n$, then

Extension of the sharp HLS on compact Riemannian manifold. There is a best constant $N(M^n, g, p, 2) > 0$, such that

$$\left|\int_{M^n}\int_{M^n} f(x)G_y(x)h(y)dV_x dV_y\right| \le N(M^n, g, p, 2)||f||_p||h||_p \tag{1.16}$$

holds for all $f \in L^p(M^n)$, $h \in L^p(M^n)$, where $p = 2n/(n+2)$. Moreover, $N(M^n, g, p, 2) \ge N(\mathbb{S}^n, g_0, p, 2)$ where $(\mathbb{S}^n, g_0)$ is the standard sphere with induced metric g_0; And for $n = 3, 4, 5$, the strict inequality holds if $G_y(x)$ has positive mass (or if (M^n, g) is not conformally equivalent to $(\mathbb{S}^n, g_0)$ by the positive mass theorem of Schoen and Yau [43]). It turns out that (1.16) is equivalent to the sharp Sobolev inequality on general compact manifold (we thank X. Wang for verifying this fact with the second named author in [47]). The general extension of sharp HLS inequality on compact manifolds, in particular for $\alpha \ne 2$, is not clear.

Let $(M^n, \partial M, g)$ $(n \ge 3)$ be a Riemannian manifold with smooth boundary ∂M. For simplicity, we consider a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, whose mean curvature function is h. Let $F_y(x)$ be the Green's function to the conformal Lapacian operator:

$$\begin{cases} -\Delta F_y(x) = 0, \ x \in \Omega, \\ \frac{\partial F_y}{\partial \nu} + \frac{n-2}{2} h F_y = \delta_y, \ x \in \partial\Omega \end{cases}$$

with pole y on the boundary, then
Extension of the sharp HLS on manifold with boundary. There is a best constant $N_b(\Omega, p, 2) > 0$, such that

$$|\int_\Omega \int_{\partial\Omega} f(y) F_y(x) g(x) dS_y dV_x| \le N_b(\Omega, p, 2) ||f||_{L^p(\partial\Omega)} ||g||_{L^t(\Omega)} \tag{1.17}$$

holds for all $f \in L^p(\partial\Omega)$, $g \in L^t(\Omega)$, where $p = 2(n-1)/n$, and $t = 2n/(n+2)$. Moreover, $N_b(\Omega, p, 2) \ge N_b(B_1(0), p, 2)$ where $B_1(0)$ is the unit ball center at the origin; And for $n = 3, 4, 5$, the strict inequality holds if Ω is not conformal to $B_1(0)$.

Now, for a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$, let $K_y(x) = n(n-2)\omega_n F_y(x)$, and define the extension operator

$$E_{2,\Omega} f(x) = \int_{\partial\Omega} f(y) K_y(x) dS_y, \ \forall\, x \in \Omega. \tag{1.18}$$

We shall be able to show that, there is a best constant $\hat{C}_\Omega(n, 2, p)$, such that

$$\|E_{2,\Omega} f\|_{L^q(\Omega)} \le \hat{C}_\Omega(n, 2, p) \|f\|_{L^p(\partial\Omega)}, \tag{1.19}$$

and $C_e(n, 2, p) \le \hat{C}_\Omega(n, 2, p)$. Moreover, for $n = 3, 4, 5$, the equality holds if and only if Ω is a ball. We believe that for $\alpha = 2$, sharp inequality (1.17) and (1.19) are equivalent to the sharp trace inequality proved in Escobar [18]. The general extension of sharp HLS on compact manifolds with boundary for $\alpha \ne 2$ is not clear.

The classification results in Theorem 1.4 are obtained for continuous functions, based on some basic calculus lemmas initially used in Li and Zhu [36] for C^1 functions. Even though the extremal functions are weak solutions to the integral equations, similar argument to the proof of Brezis and Kato's lemma implies that all weak solutions are indeed smooth. Details on the regularity of solutions are given in Section 4. Interestingly, the same calculus lemmas in Li and Zhu [36] for C^1 functions were later proved to be true for all continuous functions by Li and Nirenberg (see Appendix B in [34]), and eventually it is proved to be true for all finite, non-negative measures in $\mathbb{R}^n$ by Frank and Lieb [20]. In [20], the invariant property is used to derive the absolute continuity for the measure with respect to Lebesgue measure, which is different to our regularity argument.

For $\alpha = n$, the modern folklore inequality is obtained via taking the limit of the power, see Corollary 5.4 below. The other case $\alpha > n$ will be addressed in our future paper [16], after we explain how to extend the classical HLS inequality (1.1) for negative $\lambda = n - \alpha$ in [15]. As we pointed out at the beginning, this is another direction for extending the classical sharp HLS inequality. We need to point out that sharp Sobolev type inequalities with negative exponent on the standard sphere $\mathbb{S}^n$ did appear in, for example, Ai, Chou and Wei [1], Yang and Zhu [48], Hang and Yang [26], and Ni and Zhu [41]. See also our recent work [14] for equations with negative exponent.

Our results are closely related to the study of fractional Lapalacian operator in the whole space [8], fractional Yamabe problem [9, 22, 23, 25, 33, 34], fractional prescribing curvature problem on $\mathbb{S}^n$ [31, 32], and to the study of Yamabe and prescribing curvature problems with boundary, see, e. g. [17, 18, 19, 36, 37, 25].

Our proof of sharp integral inequalities is essentially along the line of the classical paper by Lieb [39]. Our approach to the classification of extremal function is similar to that in [36, 12, 27] and [34] in spirit, but quite different in details. For instance, we use the method of moving sphere for a system of equations to classify the extremal functions.

We shall first prove Theorem 1.1 in Section 2. The inequalities are proved along the line of the proof for the classical HLS inequality (see, e.g. Stein [45]). The sharp inequalities and the existences of extremal functions are obtained via the arguments based on symmetrization. The classification of extremal functions via the method of moving sphere is given in Section 3. In Section 4, we establish the regularity properties for solutions to the integral equations use the argument similar to the proof of Brezis and Kato's lemma [6]. In the final section, we discuss the similar inequalities in a ball, limit case, and the relation between our operators and fractional Lapalacian operators in the whole space. As a byproduct of our classification argument, we also obtain the nonexistence result (Theorem 1.5) for a subcritical integral equation system, though it is still an open problem to find the extremal function for inequality (1.8) if $p \neq 2(n-1)/(n+\alpha-2)$ (corresponding to the case $p \neq 2n/(n+\alpha)$ for HLS in whole space).

2. The sharp Hardy-Littlewood-Sobolev inequality on the upper half space

In this section, we shall establish the sharp HLS inequalities on the upper half space. We first establish the following Young inequality on the upper half space $\mathbb{R}^n_+$.

Lemma 2.1. *Suppose that $p, q, r \in [1, \infty]$ are three parameters satisfying $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. For any $h \in L^p(\partial\mathbb{R}^n_+), g \in L^q(\mathbb{R}^n_+)$, define*

$$g * h(x) = \int_{\partial\mathbb{R}^n_+} g(x-y)h(y)dy, \ \forall\, x \in \mathbb{R}^n_+.$$

Let $\tilde{g}(x') = g(x', 0)$ be the restriction of $g(x)$ on the boundary of the upper half space $\mathbb{R}^n_+$. If $|g(x', x_n)| \leq |\tilde{g}(x')|$ for all $x' \in \partial\mathbb{R}^n_+$ and $x_n > 0$, then

$$\|g * h\|_{L^r(\mathbb{R}^n_+)} \leq \|h\|_{L^p(\partial\mathbb{R}^n_+)} \|g\|^{q/r}_{L^q(\mathbb{R}^n_+)} \|\tilde{g}\|^{1-q/r}_{L^q(\partial\mathbb{R}^n_+)}. \tag{2.1}$$

Proof. If $r = \infty$, then $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder inequality and the translation invariance of Lebesgue measure, we have

$$\begin{aligned} \left|\int_{\partial\mathbb{R}^n_+} g(x'-y, x_n)h(y)dy\right| &\leq \int_{\partial\mathbb{R}^n_+} |\tilde{g}(x'-y)||h(y)|dy \\ &\leq \|\tilde{g}\|_{L^q(\partial\mathbb{R}^n_+)}\|h\|_{L^p(\partial\mathbb{R}^n_+)}. \end{aligned}$$

Hence,

$$\|g * h\|_{L^\infty(\mathbb{R}^n_+)} \leq \|\tilde{g}\|_{L^q(\partial\mathbb{R}^n_+)}\|h\|_{L^p(\partial\mathbb{R}^n_+)}.$$

Next, we consider the case $1 \le r < \infty$. For any $a, b \in [0,1]$, $p_1, p_2 \in [0, \infty]$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{r} = 1$, and $x \in \mathbb{R}^n_+$, we have

$$\begin{aligned}|g*h(x)| &\le \int_{\partial\mathbb{R}^n_+} |g(x-y)|^{1-a}|h(y)|^{1-b}|g(x'-y,x_n)|^a|h(y)|^b dy\\ &\le \int_{\partial\mathbb{R}^n_+} |g(x-y)|^{1-a}|h(y)|^{1-b}|\tilde{g}(x'-y)|^a|h(y)|^b dy\\ &\le \Big(\int_{\partial\mathbb{R}^n_+} |g(x-y)|^{(1-a)r}|h(y)|^{(1-b)r}dy\Big)^{\frac{1}{r}}\|\tilde{g}\|^a_{L^{ap_1}(\partial\mathbb{R}^n_+)}\|h\|^b_{L^{bp_2}(\partial\mathbb{R}^n_+)}.\end{aligned}$$

Taking the r^{th} power of the above inequality and integrating on $\mathbb{R}^n_+$, we have

$$\begin{aligned}\int_{\mathbb{R}^n_+}|g*h(x)|^r dx &\le \|\tilde{g}\|^{ar}_{L^{ap_1}(\partial\mathbb{R}^n_+)}\|h\|^{br}_{L^{bp_2}(\partial\mathbb{R}^n_+)}\cdot\int_{\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+}|g(x-y)|^{(1-a)r}|h(y)|^{(1-b)r}dydx\\ &= \|\tilde{g}\|^{ar}_{L^{ap_1}(\partial\mathbb{R}^n_+)}\|h\|^{br}_{L^{bp_2}(\partial\mathbb{R}^n_+)}\cdot\int_{\partial\mathbb{R}^n_+}|h(y)|^{(1-b)r}\Big(\int_{\mathbb{R}^n_+}|g(x-y)|^{(1-a)r}dx\Big)dy\\ &= \|h\|^{(1-b)r}_{L^{(1-b)r}(\partial\mathbb{R}^n_+)}\|g\|^{(1-a)r}_{L^{(1-a)r}(\mathbb{R}^n_+)}\|\tilde{g}\|^{ar}_{L^{ap_1}(\partial\mathbb{R}^n_+)}\|h\|^{br}_{L^{bp_2}(\partial\mathbb{R}^n_+)}.\end{aligned}$$

Now, choosing $p = (1-b)r = bp_2, q = (1-a)r = ap_1$, we have inequality (2.1).

To complete the proof, we still need to show that p, q and r are arbitrary indices in $[1,\infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. Since $p = (1-b)r = bp_2, q = (1-a)r = ap_1$, we have $a = 1 - \frac{q}{r}, b = 1 - \frac{p}{r}$, and $p_1 = \frac{q}{a}, p_2 = \frac{p}{b}$. We can directly verify that

$$1 = \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{r} = \frac{1}{q} + \frac{1}{p} - \frac{1}{r}.$$

Conversely, if $p, q, r \in [1,\infty)$ are given and satisfy $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, then $a = 1 - \frac{q}{r} \ge 0$ and $b = 1 - \frac{p}{r} \ge 0$. It is obvious that both $a, b \le 1$. Also, for $p_1 = \frac{q}{a}, p_2 = \frac{p}{b}$, it is easy to check that

$$\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{r} = 1.$$

□

We are now ready to prove the inequality in Theorem 1.1 with a constant (which may not be sharp).

Proof of Theorem 1.1.

For $\alpha \in (1,n)$, $p \in [1, \frac{n-1}{\alpha-1})$ and q given by $\frac{1}{q} = (\frac{1}{p} - \frac{\alpha-1}{n-1}) \cdot \frac{n-1}{n}$, we first prove

$$\|E_\alpha f\|_{L^q_W(\mathbb{R}^n_+)} \le C(n,\alpha,p)\|f\|_{L^p(\partial\mathbb{R}^n_+)} \tag{2.2}$$

for some constant $C(n,\alpha,p)$. That is, we need to show that there is a constant $C(n,\alpha,p) > 0$, such that

$$meas\{x : |E_\alpha f(x)| > \lambda\} \le \Big(C(n,\alpha,p)\frac{\|f\|_{L^p(\partial\mathbb{R}^n_+)}}{\lambda}\Big)^q, \ \forall f \in L^p(\partial\mathbb{R}^n_+), \ \forall\ \lambda > 0. \tag{2.3}$$

Note that inequality (2.2) implies, via the Marcinkiewicz interpolation [45, 42], that

$$\|E_\alpha f\|_{L^q(\mathbb{R}^n_+)} \le C_1(n,\alpha,p)\|f\|_{L^p(\partial\mathbb{R}^n_+)},$$

or even slight stronger inequality

$$\|E_\alpha f\|_{L^q(\mathbb{R}^n_+)} \le C_2(n,\alpha,p)\|f\|_{L^{p,q}(\partial\mathbb{R}^n_+)} \tag{2.4}$$

for $p \in (1, \frac{n-1}{\alpha-1})$ and q given by $\frac{1}{q} = (\frac{1}{p} - \frac{\alpha-1}{n-1}) \cdot \frac{n-1}{n}$.

Let X be either $\mathbb{R}^n_+$ or $\partial\mathbb{R}^n_+$. Recall: for a given measurable function $f(x)$ on X and $1 \le p < \infty$, the weak L^p norm of $f(x)$ is defined by

$$\|f\|_{L^p_W(X)} = \inf\{A > 0 \,|\, meas\{x \in X \mid f(x) > t\} \cdot t^p \le A^p\},$$

and $L^p_W(X) = \{f \,|\, f$ is a measurable function, and $\|f\|_{L^p_W(X)} < \infty\}$. More generally, for $1 < p < \infty, 1 \le q < \infty$, the Lorentz $L^{p.q}(X)$ norm is given by

$$\|f\|_{L^{p,q}(X)} = \Big[\int_0^\infty (x^{\frac{1}{p}} f^{**}(x))^q \frac{dx}{x}\Big]^{\frac{1}{q}}.$$

where $f^{**} = \frac{1}{x}\int_0^x f^*(t)dt$, f^* is the nonnegative nonincreasing rearrangement of $|f|$ (see, O'Neail [42] P_{136}). For $1 \le p \le \infty, q = \infty$,

$$\|f\|_{L^{p,\infty}(X)} = \sup_{x>0} x^{\frac{1}{p}} f^{**}(x),$$

and the Lorentz space $L^{p,q}(X) = \{f \,|\, f$ is a measurable function, and $\|f\|_{L^{p,q}(X)} < \infty\}$. It is known that $L^p_W(X) = L^{p,\infty}(X)$ is a special case of such space. See also, Stein [45].

For any $r > 0$, define

$$E^1_{\alpha,r} f(x) = \int_{\partial\mathbb{R}^n_+, |y-x| \le r} \frac{f(y)}{|x-y|^{n-\alpha}} dy,$$

and

$$E^2_{\alpha,r} f(x) = \int_{\partial\mathbb{R}^n_+, |y-x| \ge r} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Then for any $\lambda > 0$,

$$meas\{x : |E_\alpha f(x)| > 2\lambda\} \le meas\{x : |E^1_{\alpha,r} f(x)| > \lambda\} + meas\{x : |E^2_{\alpha,r} f(x)| > \lambda\}. \tag{2.5}$$

It is enough to prove inequality (2.3) with 2λ in place of λ in the left side of the inequality. We can further assume $\|f\|_{L^p(\partial\mathbb{R}^n_+)} = 1$.

Since $|x - y| \ge |x' - y|$ for any $x = (x', x_n) \in \mathbb{R}^n_+$ and $y \in \partial\mathbb{R}^n_+$, using Young inequality (2.1) (with $q = 1$), we have

$$\begin{aligned} \|E^1_{\alpha,r} f\|_{L^p(\mathbb{R}^n_+)} &\le \Big(\int_{\mathbb{R}^n_+} \frac{\chi_r(|x-y|)}{|x-y|^{n-\alpha}} dx\Big)^{\frac{1}{p}} \Big(\int_{\partial\mathbb{R}^n_+} \frac{\chi_r(|x'-y|)}{|x'-y|^{n-\alpha}} dy\Big)^{\frac{p-1}{p}} \|f\|_{L^p(\partial\mathbb{R}^n_+)} \\ &=: D_1 D_2, \end{aligned}$$

where $\chi_r(x) = 1$ for $|x| \le r$ and $\chi_r(x) = 0$ for $|x| > r$, and

$$\begin{aligned} D_1 &= \Big(\int_{B^+_r(y)} \frac{1}{|x-y|^{n-\alpha}} dx\Big)^{\frac{1}{p}} = C_1(n,\alpha,p) r^{\frac{\alpha}{p}}, \\ D_2 &= \Big(\int_{B^{n-1}_r(x')} \frac{1}{|x'-y|^{n-\alpha}} dy\Big)^{\frac{p-1}{p}} = C_2(n,\alpha,p) r^{\frac{(\alpha-1)(p-1)}{p}}. \end{aligned}$$

We use the fact that $\alpha > 1$ in the computation for D_2. It follows that

$$meas\{x : |E^1_{\alpha,r} f| > \lambda\} \le \frac{\|E^1_{\alpha,r} f\|^p_{L^p(\mathbb{R}^n_+)}}{\lambda^p} \le \frac{C_3(n,\alpha,p) r^{\alpha p-(p-1)}}{\lambda^p}. \tag{2.6}$$

On the other hand,

$$\begin{aligned}\|E^2_{\alpha,r}f\|_{L^\infty(\mathbb{R}^n_+)} &\leq \int_{\partial\mathbb{R}^n_+}\frac{1-\chi_r(|x'-y|)}{|x'-y|^{n-\alpha}}|f(y)|dy\\ &\leq (\int_{\partial\mathbb{R}^n_+}(\frac{1-\chi_r(|x'-y|)}{|x'-y|^{n-\alpha}})^{p'}dy)^{\frac{1}{p'}}\|f\|_{L^p(\partial\mathbb{R}^n_+)}\\ &=: D_3,\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. It is easy to see (note: $p<(n-1)/(\alpha-1)$) that

$$D_3 = (\int_{\partial\mathbb{R}^n_+\setminus B^{n-1}_r(x')}(\frac{1}{|x'-y|^{n-\alpha}})^{p'}dy)^{\frac{1}{p'}} = C_4(n,\alpha,p)r^{\frac{(n-1)-(n-\alpha)p'}{p'}}.$$

If we choose $r=(C_4^{-1}(n,\alpha,p)\lambda)^{\frac{p'}{(n-1)-(n-\alpha)p'}}$, thus $\lambda=C_4(n,\alpha,p)r^{\frac{(n-1)-(n-\alpha)p'}{p'}}$. It follows that $\|E^2_{\alpha,r}f\|_{L^\infty(\mathbb{R}^n_+)}\leq\lambda$, thus

$$meas\{x:|E^2_{\alpha,r}f|>\lambda\}=0.$$

Bringing the above and (2.6) into (2.5), we have

$$\begin{aligned}meas\{x:|E_\alpha f|>2\lambda\} &\leq meas\{x:|E^1_{\alpha,r}f|>\lambda\}\leq\frac{C_3(n,\alpha)r^{\alpha p-(p-1)}}{\lambda^p}\\ &= C_5(n,\alpha)\lambda^{\frac{p'[\alpha p-(p-1)]}{(n-1)-(n-\alpha)p'}-p},\end{aligned}$$

where

$$\frac{p'[\alpha p-(p-1)]}{(n-1)-(n-\alpha)p'}-p = -\frac{np}{(n-1)-(\alpha-1)p}.$$

Let

$$q=\frac{np}{(n-1)-(\alpha-1)p},$$

that is,

$$\frac{1}{q}=\frac{n-1}{n}(\frac{1}{p}-\frac{\alpha-1}{n-1}),$$

we then obtain (2.3).

The sharp constant to inequality (1.8) is classified by

$$C_e(n,\alpha,p)=\sup\{\|E_\alpha f\|_{L^q(\mathbb{R}^n_+)}\ :\ f\in L^p(\partial\mathbb{R}^n_+),\ \|f\|_{L^p(\partial\mathbb{R}^n_+)}=1\}.$$

Using symmetrization argument, we will show that the above supreme is attained by a radially symmetric (with respect to some point) function. Moreover, we will show that all extremal functions for inequality (1.8) are radially symmetric with respect to some point.

For any measurable function $u(x)$ on $\mathbb{R}^n$ vanishing at infinity, we can define its radially symmetric, non-increasing rearrangement function u^*. $u^*(x)$ is a nonnegative lower-semicontinuous function and has the same distribution as $|u|$. Moreover, for $1\leq p\leq\infty$, the following inequality holds

$$\|u*v\|_{L^p(\mathbb{R}^n)}\leq\|u^**v^*\|_{L^p(\mathbb{R}^n)}, \tag{2.7}$$

where $u*v(x)=\int_{\mathbb{R}^n}u(y)v(x-y)dy$ is the convolution product. See, for example, Brascamp, Lieb and Luttinger [3].

Let $\{f_j\}_{j=1}^\infty\in C_0^\infty(\partial\mathbb{R}^n_+)$ be a nonnegative maximizing sequence with $\|f_j\|_{L^p(\partial\mathbb{R}^n_+)}=1$, f_j^* be the rearrangement of f_j.

Since

$$\|f_j^*\|_{L^p(\partial\mathbb{R}^n_+)} = \|f_j\|_{L^p(\partial\mathbb{R}^n_+)} = 1,$$

and by (2.7),

$$\begin{aligned}\|E_\alpha(f_j)\|^q_{L^q(\mathbb{R}^n_+)} &= \int_0^\infty \int_{\partial\mathbb{R}^n_+} \Big(\int_{\partial\mathbb{R}^n_+} \frac{f_j(y)}{(|x'-y|^2+x_n^2)^{\frac{n-\alpha}{2}}}dy\Big)^q dx'dx_n \\ &\le \int_0^\infty \int_{\partial\mathbb{R}^n_+} \Big(\int_{\partial\mathbb{R}^n_+} \frac{f_j^*(y)}{(|x'-y|^2+x_n^2)^{\frac{n-\alpha}{2}}}dy\Big)^q dx'dx_n \\ &= \|E_\alpha(f_j^*)\|^q_{L^q(\mathbb{R}^n_+)},\end{aligned}$$

we know that $\{f_j^*\}_{j=1}^\infty$ is also a maximizing sequence. Without loss of generality, we can assume that this is a sequence of nonnegative radially symmetric and non-increasing functions. To avoid that f_j may converge to a trivial function, we need to modify the sequence further.

For convenience, denote $e_1 = (e_1', 0) = (1, 0, \cdots, 0, 0) \in \mathbb{R}^n$, and

$$a_j := sup_{\lambda>0}\lambda^{-\frac{n-1}{p}} f_j(\frac{e_1'}{\lambda}).$$

Note that for $y \in \partial\mathbb{R}^n_+ \setminus \{0\}$,

$$0 \le f_j(y) = f_j(|y|e_1') = |y|^{-\frac{n-1}{p}}|y|^{\frac{n-1}{p}} f_j(|y|e_1') \le a_j|y|^{-\frac{n-1}{p}},$$

and

$$\|f_j\|_{L^{p,\infty}(\partial\mathbb{R}^n_+)} \le Ca_j.$$

Thus, for $\frac{1}{q} = (\frac{1}{p} - \frac{\alpha-1}{n-1})\cdot\frac{n-1}{n}$, by (2.4) we have

$$\begin{aligned}\|E_\alpha(f_j)\|_{L^q(\mathbb{R}^n_+)} &\le C(n,\alpha,p)\|f_j\|_{L^{p,q}(\partial\mathbb{R}^n_+)} \\ &\le C(n,\alpha,p)\|f_j\|^{\frac{n-1-(\alpha-1)p}{n}}_{L^{p,p}(\partial\mathbb{R}^n_+)}\|f_j\|^{\frac{1+(\alpha-1)p}{n}}_{L^{p,\infty}(\partial\mathbb{R}^n_+)} \\ &\le C(n,\alpha,p)a_j^{\frac{1+(\alpha-1)p}{n}}.\end{aligned}$$

This implies $a_j \ge C(n,\alpha,p) > 0$. Define $f_j^\lambda(x) = \lambda^{-\frac{n-1}{p}} f(\frac{x}{\lambda})$. Thus, we may choose λ_j such that

$$f_j^{\lambda_j}(e_1') = \lambda_j{}^{-\frac{n-1}{p}} f(\frac{e_1'}{\lambda_j}) \ge C(n,\alpha,p) > 0.$$

Also, it is easy to verify that

$$\|f_j^{\lambda_j}\|_{L^p(\partial\mathbb{R}^n_+)} = \|f_j\|_{L^p(\partial\mathbb{R}^n_+)}, \quad \text{and} \quad \|E_\alpha(f_j^{\lambda_j})\|_{L^q(\mathbb{R}^n_+)} = \|E_\alpha(f_j)\|_{L^q(\mathbb{R}^n_+)}.$$

Thus, $\{f_j^{\lambda_j}\}_{j=1}^\infty$ is also a maximizing sequence. Therefore, we can further assume that the nonnegative and radially non-increasing maximizing sequence $\{f_j\}_{j=1}^\infty$ with $\|f_j\|_{L^p(\partial\mathbb{R}^n_+)} = 1$ also satisfies $f_j(e_1') \ge C(n,\alpha,p) > 0$ for all $j \ge 1$. It follows that

$$|f_j(y)| \le \omega_{n-1}^{-\frac{1}{p}}|y|^{-\frac{n-1}{p}}, \qquad \forall y \in \partial\mathbb{R}^n_+. \tag{2.8}$$

Up to a subsequence, we can find a nonnegative, radially nonincreasing function f such that $f_j \to f$ a.e.. Hence we conclude that $f(y) \ge C(n,\alpha,p) > 0$ for

$|y| \le 1, f_j \rightharpoonup f$ in $L^p(\partial\mathbb{R}^n_+)$, and $\|f\|_{L^p(\partial\mathbb{R}^n_+)} \le 1$. From Brezis and Lieb's Lemma [5], we see

$$\int_{\partial\mathbb{R}^n_+} \big||f_j|^p - |f|^p - |f_j - f|^p\big| dy \to 0.$$

So

$$\begin{aligned}\|f_j - f\|^p_{L^p(\partial\mathbb{R}^n_+)} &= \|f_j\|^p_{L^p(\partial\mathbb{R}^n_+)} - \|f\|^p_{L^p(\partial\mathbb{R}^n_+)} + o(1)\\ &= 1 - \|f\|^p_{L^p(\partial\mathbb{R}^n_+)} + o(1).\end{aligned} \tag{2.9}$$

Note $\alpha \in (1, n)$. Also from (2.8) we have: for any fixed $x \ne 0$, and $|y| > 2|x|$,

$$\Big|\frac{f_j(y)}{|x-y|^{n-\alpha}} - \frac{f(y)}{|x-y|^{n-\alpha}}\Big| \le \frac{C}{|y|^{n-\alpha+(n-1)/p}}.$$

Thus $(E_\alpha f_j)(x) \to (E_\alpha f)(x)$ for almost all $x \in \mathbb{R}^n_+$ by dominated convergence theory. It follows that

$$\begin{aligned}C_e(n,\alpha,p) + o(1) &= \|E_\alpha f_j\|^q_{L^q(\mathbb{R}^n_+)}\\ &= \|E_\alpha f\|^q_{L^q(\mathbb{R}^n_+)} + \|E_\alpha(f_j - f)\|^q_{L^q(\mathbb{R}^n_+)} + o(1)\\ &\le C_e^q(n,\alpha,p)\|f\|^q_{L^p(\partial\mathbb{R}^n_+)} + C_e^q(n,\alpha,p)\|f_j - f\|^q_{L^p(\partial\mathbb{R}^n_+)} + o(1).\end{aligned}$$

So

$$1 \le \|f\|^q_{L^p(\partial\mathbb{R}^n_+)} + \|f_j - f\|^q_{L^p(\partial\mathbb{R}^n_+)} + o(1). \tag{2.10}$$

Combining (2.9) and (2.10) and letting $j \to \infty$, we obtain

$$1 \le \|f\|^q_{L^p(\partial\mathbb{R}^n_+)} + \big(1 - \|f\|^p_{L^p(\partial\mathbb{R}^n_+)}\big)^{\frac{q}{p}}.$$

From the fact that $q > p$ and $f \not\equiv 0$, we have $\|f\|_{L^p(\partial\mathbb{R}^n_+)} = 1$. Hence $f_j \to f$ in $L^p(\partial\mathbb{R}^n_+)$ and f is a maximizer. This shows the existence of an extremal function to the sharp inequality (1.8).

Next we show the radial symmetry and monotonicity hold for all extremal functions. Assume that $f \in L^p(\partial\mathbb{R}^n_+)$ is a maximizer, so is $|f|$. Thus $\|E_\alpha f\|_{L^q(\mathbb{R}^n_+)} = \|E_\alpha |f|\|_{L^q(\mathbb{R}^n_+)}$, which implies $f \ge 0$ or $f \le 0$. Without loss of generality, we only consider the case of $f \ge 0$. Then the Euler-Lagrange equation for $f(x)$ is, up to a constant multiplier, given by equation (1.13)

$$f^{p-1}(y) = \int_{\mathbb{R}^n_+} \frac{(E_\alpha f(x))^{q-1}}{|x-y|^{n-\alpha}} dx, \qquad \forall\, y \in \partial\mathbb{R}^n_+.$$

On the other hand, (see, e.g. [38] P_{103}), if u is nonnegative, radially symmetric, and strictly decreasing in the radial direction, v is nonnegative, $1 < p < \infty$ and

$$\|u * v\|_{L^p(\mathbb{R}^n)} = \|u * v^*\|_{L^p(\mathbb{R}^n)} < \infty,$$

then $v(x) = v^*(x - x_0)$ for some $x_0 \in \mathbb{R}^n$. It follows from this fact and the Euler-Lagrange equation satisfied by $f(y)$, that $f(y) = f^*(y - y_0) = f^*(|y - y_0|)$ for some $y_0 \in \partial\mathbb{R}^n_+$, where $f^*(r)$ is decreasing in r.

3. Classification of positive solutions for integral equations

In this section, we classify all nonnegative solutions to integral equation (1.13) for $p = 2(n-1)/(n+\alpha-2)$ and $q = 2n/(n-\alpha)$.

Let f be a nonnegative function satisfying (1.13). Define $u(y) = f^{p-1}(y), v(x) = E_\alpha f(x)$, $\theta = \frac{1}{p-1}$, and $\kappa = q-1$. Then the single equation (1.13) can be rewritten as an integral system

$$\begin{cases} u(y) = \int_{\mathbb{R}^n_+} \frac{v^\kappa(x)}{|x-y|^{n-\alpha}} dx, & y \in \partial\mathbb{R}^n_+, \\ v(x) = \int_{\partial\mathbb{R}^n_+} \frac{u^\theta(y)}{|x-y|^{n-\alpha}} dy, & x \in \mathbb{R}^n_+ \end{cases} \tag{3.1}$$

with $\frac{1}{\kappa+1} = \frac{n-1}{n}\big(\frac{n-\alpha}{n-1} - \frac{1}{\theta+1}\big)$. Note: if $f \in L^p_{loc}(\partial\mathbb{R}^n_+)$, then $u \in L^{\theta+1}_{loc}(\partial\mathbb{R}^n_+)$. In next section, we will show that if (u,v) is a pair of nonnegative solutions to system (3.1) with $u \in L^{\theta+1}_{loc}(\partial\mathbb{R}^n_+)$, then $u \in C^\infty(\partial\mathbb{R}^n_+)$ and $v \in C^\infty(\overline{\mathbb{R}^n_+})$. From now on in this section, we assume both u, v are smooth functions. Theorem 1.4 follows from the following theorem.

Theorem 3.1. *Let (u,v) be a pair of positive smooth solutions to system* (3.1). *For $1 < \alpha < n$, if*

$$\theta = \frac{n+\alpha-2}{n-\alpha}, \kappa = \frac{n+\alpha}{n-\alpha},$$

then u, v must be the following forms on $\partial\mathbb{R}^n_+$:

$$u(\xi) = c_1\big(\frac{1}{|\xi-\xi_0|^2+d^2}\big)^{\frac{n-\alpha}{2}},$$
$$v(\xi,0) = c_2\big(\frac{1}{|\xi-\xi_0|^2+d^2}\big)^{\frac{n-\alpha}{2}},$$

where c_1, c_2, $d > 0, \xi_0 \in \partial\mathbb{R}^n_+$.

We prove the above theorem via the method of moving sphere, which is introduced by Li and Zhu in [36].

For $R > 0$, denote

$$B_R(x) = \{y \in \mathbb{R}^n \,:\, |y-x| < R, x \in \mathbb{R}^n\},\ B^{n-1}_R(x) = \{y \in \partial\mathbb{R}^n_+ \,:\, |y-x| < R, x \in \partial\mathbb{R}^n_+\},$$

$$B^+_R(x) = \{y = (y_1, y_2, \cdots, y_n) \in B_R(x) \,:\, y_n > 0, x \in \partial\mathbb{R}^n_+\},$$

$$\Sigma^n_{x,R} = \mathbb{R}^n_+ \backslash \overline{B^+_R(x)},\ \Sigma^{n-1}_{x,R} = \partial\mathbb{R}^n_+ \backslash \overline{B^{n-1}_R(x)}.$$

For $x \in \partial\mathbb{R}^n_+$ and $\lambda > 0$, we define the following transform:

$$\omega_{x,\lambda}(\xi) = \big(\frac{\lambda}{|\xi-x|}\big)^{n-\alpha}\omega(\xi^{x,\lambda}), \quad \xi \in \overline{\mathbb{R}^n_+} \backslash \{x\},$$

where

$$\xi^{x,\lambda} = x + \frac{\lambda^2(\xi-x)}{|\xi-x|^2}$$

is the Kelvin transformation of ξ with respect to $B^+_\lambda(x)$. Also we write $\omega^k_{x,\lambda}(\xi) := (\omega_{x,\lambda}(\xi))^k$.

First we have the following lemma.

Lemma 3.2. *Let $1 < \alpha < n, 0 < \theta, \kappa < \infty$ and (u, v) be a pair of positive solutions to system (3.1). Then, for any $x \in \partial\mathbb{R}^n_+$,*

$$u_{x,\lambda}(\xi) = \int_{\mathbb{R}^n_+} \frac{v^\kappa_{x,\lambda}(\eta)}{|\xi - \eta|^{n-\alpha}} \big(\frac{\lambda}{|\eta - x|}\big)^{\tau_1} d\eta, \quad \forall\, \xi \in \partial\mathbb{R}^n_+, \tag{3.2}$$

$$v_{x,\lambda}(\eta) = \int_{\partial\mathbb{R}^n_+} \frac{u^\theta_{x,\lambda}(\xi)}{|\xi - \eta|^{n-\alpha}} \big(\frac{\lambda}{|\xi - x|}\big)^{\tau_2} d\xi, \quad \forall\, \eta \in \mathbb{R}^n_+, \tag{3.3}$$

where $\tau_1 = n + \alpha - \kappa(n - \alpha), \tau_2 = n + \alpha - 2 - \theta(n - \alpha)$. Moreover,

$$u_{x,\lambda}(\xi) - u(\xi) = \int_{\Sigma^n_{x,\lambda}} P(x, \lambda; \xi, \eta)\big[\big(\frac{\lambda}{|\eta - x|}\big)^{\tau_1} v^\kappa_{x,\lambda}(\eta) - v^\kappa(\eta)\big] d\eta, \tag{3.4}$$

$$v_{x,\lambda}(\eta) - v(\eta) = \int_{\Sigma^{n-1}_{x,\lambda}} P(x, \lambda; \eta, \xi)\big[\big(\frac{\lambda}{|\xi - x|}\big)^{\tau_2} u^\theta_{x,\lambda}(\xi) - u^\theta(\xi)\big] d\xi, \tag{3.5}$$

where

$$P(x, \lambda; \xi, \eta) = \frac{1}{|\xi - \eta|^{n-\alpha}} - \big(\frac{\lambda}{|\xi - x|}\big)^{n-\alpha} \frac{1}{|\xi^{x,\lambda} - \eta|^{n-\alpha}},$$

and

$$P(x, \lambda; \xi, \eta) > 0, \ \textit{for} \ \forall\, \xi \in \Sigma^{n-1}_{x,\lambda},\ \eta \in \Sigma^n_{x,\lambda}, \lambda > 0.$$

The proof of Lemma 3.2 is similar to that in [34]. Similar computation will also be used to derive an analog inequality (1.14) on a ball from inequality (1.8) in Section 5.1.

Proof. For any $x \in \partial\mathbb{R}^n_+, \eta \in \mathbb{R}^n_+$ and $\lambda > 0$, let

$$y = \eta^{x,\lambda} = x + \frac{\lambda^2(\eta - x)}{|\eta - x|^2}.$$

The n dimensional volume forms in y variable and η variable are related by

$$dy = \big(\frac{\lambda}{|\eta - x|}\big)^{2n} d\eta.$$

To simplify the calculations, we write

$$A^+(\xi^{x,\lambda}) = \int_{\Sigma^n_{x,\lambda}} \frac{v^\kappa(y)}{|\xi^{x,\lambda} - y|^{n-\alpha}} dy, \quad A^-(\xi^{x,\lambda}) = \int_{B^+_\lambda(x)} \frac{v^\kappa(y)}{|\xi^{x,\lambda} - y|^{n-\alpha}} dy.$$

Thus, from (3.1) we can rewrite u as follows:

$$u(\xi^{x,\lambda}) = A^+(\xi^{x,\lambda}) + A^-(\xi^{x,\lambda})$$

for $x, \xi \in \partial\mathbb{R}^n_+$. Direct calculation yields

$$\begin{aligned} A^+(\xi^{x,\lambda}) &= \int_{\Sigma^n_{x,\lambda}} \frac{v^\kappa(y)}{|\xi^{x,\lambda} - y|^{n-\alpha}} dy = \int_{B^+_\lambda(x)} \frac{v^\kappa(\eta^{x,\lambda})}{|\xi^{x,\lambda} - \eta^{x,\lambda}|^{n-\alpha}} \big(\frac{\lambda}{|\eta - x|}\big)^{2n} d\eta \\ &= \int_{B^+_\lambda(x)} \frac{v^\kappa_{x,\lambda}(\eta)}{|\xi^{x,\lambda} - \eta^{x,\lambda}|^{n-\alpha}} \big(\frac{\lambda}{|\eta - x|}\big)^{2n-\kappa(n-\alpha)} d\eta. \end{aligned}$$

Note that:

$$\frac{|\eta - x|}{\lambda} \frac{|\xi - x|}{\lambda} |\xi^{x,\lambda} - \eta^{x,\lambda}| = |\xi - \eta|.$$

Thus,

$$\begin{aligned}
A^{+}_{x,\lambda}(\xi) &:= \Big(\frac{\lambda}{|\xi-x|}\Big)^{n-\alpha}A^{+}(\xi^{x,\lambda})\\
&= \Big(\frac{\lambda}{|\xi-x|}\Big)^{n-\alpha}\int_{B^{+}_{\lambda}(x)}\frac{v^{\kappa}_{x,\lambda}(\eta)}{|\xi^{x,\lambda}-\eta^{x,\lambda}|^{n-\alpha}}\Big(\frac{\lambda}{|\eta-x|}\Big)^{2n-\kappa(n-\alpha)}d\eta\\
&= \int_{B^{+}_{\lambda}(x)}\frac{v^{\kappa}_{x,\lambda}(\eta)}{|\xi-\eta|^{n-\alpha}}\Big(\frac{\lambda}{|\eta-x|}\Big)^{\tau_1}d\eta.
\end{aligned}$$

Similarly, we have

$$A^{-}_{x,\lambda}(\xi) := \Big(\frac{\lambda}{|\xi-x|}\Big)^{n-\alpha}A^{-}(\xi^{x,\lambda}) = \int_{\Sigma^{n}_{x,\lambda}}\frac{v^{\kappa}_{x,\lambda}(\eta)}{|\xi-\eta|^{n-\alpha}}\Big(\frac{\lambda}{|\eta-x|}\Big)^{\tau_1}d\eta.$$

Hence,

$$u_{x,\lambda}(\xi) = A^{+}_{x,\lambda}(\xi)+A^{-}_{x,\lambda}(\xi) = \int_{\mathbb{R}^{n}_{+}}\frac{v^{\kappa}_{x,\lambda}(\eta)}{|\xi-\eta|^{n-\alpha}}\Big(\frac{\lambda}{|\eta-x|}\Big)^{\tau_1}d\eta.$$

Identity (3.2) is established.

It also follows that

$$\begin{aligned}
u_{x,\lambda}(\xi)-u(\xi) &= A^{+}_{x,\lambda}(\xi)+A^{-}_{x,\lambda}(\xi)-\big(A^{+}(\xi)+A^{-}(\xi)\big)\\
&= (A^{-}_{x,\lambda}(\xi)-A^{+}(\xi))+(A^{+}_{x,\lambda}(\xi)-A^{-}(\xi))\\
&= \int_{\Sigma^{n}_{x,\lambda}}\frac{1}{|\xi-\eta|^{n-\alpha}}\Big[\Big(\frac{\lambda}{|\eta-x|}\Big)^{\tau_1}v^{\kappa}_{x,\lambda}(\eta)-v^{\kappa}(\eta)\Big]d\eta\\
&\quad +(A^{+}_{x,\lambda}(\xi)-A^{-}(\xi)).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
A^{+}_{x,\lambda}(\xi) &= \Big(\frac{\lambda}{|\xi-x|}\Big)^{n-\alpha}A^{+}(\xi^{x,\lambda})\\
&= \Big(\frac{\lambda}{|\xi-x|}\Big)^{n-\alpha}\int_{\Sigma^{n}_{x,\lambda}}\frac{v^{\kappa}(y)}{|\xi^{x,\lambda}-y|^{n-\alpha}}dy,
\end{aligned}$$

and

$$\begin{aligned}
A^{-}(\xi) &= A^{-}((\xi^{x,\lambda})^{x,\lambda}) = \Big(\frac{\lambda}{|\xi^{x,\lambda}-x|}\Big)^{\alpha-n}A^{-}_{x,\lambda}(\xi^{x,\lambda})\\
&= \Big(\frac{\lambda}{|\xi^{x,\lambda}-x|}\Big)^{\alpha-n}\int_{\Sigma^{n}_{x,\lambda}}\frac{v^{\kappa}_{x,\lambda}(\eta)}{|\xi^{x,\lambda}-\eta|^{n-\alpha}}\Big(\frac{\lambda}{|\eta-x|}\Big)^{\tau_1}d\eta\\
&= \Big(\frac{\lambda}{|\xi-x|}\Big)^{n-\alpha}\int_{\Sigma^{n}_{x,\lambda}}\frac{v^{\kappa}_{x,\lambda}(\eta)}{|\xi^{x,\lambda}-\eta|^{n-\alpha}}\Big(\frac{\lambda}{|\eta-x|}\Big)^{\tau_1}d\eta.
\end{aligned}$$

Combining the above computations, we have identity (3.4). (3.3) and (3.5) can be obtained in the same way.

Now, for $\xi \in \Sigma_{x,\lambda}^{n-1}, \eta \in \Sigma_{x,\lambda}^{n}$ and $\lambda > 0$, we have

$$
\begin{aligned}
P(x,\lambda;\xi,\eta) &= |\xi-\eta|^{\alpha-n} - \big(\frac{|\xi-x|}{\lambda}\big)^{\alpha-n}|\xi^{x,\lambda}-\eta|^{\alpha-n} \\
&= |\xi-\eta|^{\alpha-n} - \frac{1}{\lambda^{\alpha-n}}\big(\big|\frac{\lambda^2(\xi-x)}{|\xi-x|} + |\xi-x|(x-\eta)\big|^2\big)^{\frac{\alpha-n}{2}} \\
&= |\xi-\eta|^{\alpha-n} - \frac{1}{\lambda^{\alpha-n}}\big[\lambda^4 + 2\lambda^2(\xi-x)(x-\eta) + (\xi-x)^2(x-\eta)^2\big]^{\frac{\alpha-n}{2}} \\
&> |\xi-\eta|^{\alpha-n} - \frac{1}{\lambda^{\alpha-n}}\big[(x-\eta)^4 + 2(x-\eta)^2(\xi-x)(x-\eta) + (\xi-x)^2(x-\eta)^2\big]^{\frac{\alpha-n}{2}} \\
&= |\xi-\eta|^{\alpha-n} - \big(\frac{|x-\eta|}{\lambda}\big)^{\alpha-n}\big[(x-\eta+\xi-x)^2\big]^{\frac{\alpha-n}{2}} \\
&= |\xi-\eta|^{\alpha-n}\big(1-\big(\frac{|x-\eta|}{\lambda}\big)^{\alpha-n}\big) > 0.
\end{aligned}
$$

Lemma 3.2 is proved. □

Let

$$\gamma := \frac{n+\alpha-2}{n-\alpha}, \ \ \beta := \frac{n+\alpha}{n-\alpha}.$$

It is clear in Lemma 3.2 that $\tau_1 = \tau_2 = 0$ if and only if $\theta = \gamma$ and $\kappa = \beta$. From now on in this section, we assume that $\theta = \gamma$ and $\kappa = \beta$.

Define

$$\Sigma_{x,\lambda}^{u} = \{\xi \in \Sigma_{x,\lambda}^{n-1} \,|\, u(\xi) < u_{x,\lambda}(\xi)\}, \quad \text{and} \quad \Sigma_{x,\lambda}^{v} = \{\eta \in \Sigma_{x,\lambda}^{n} \,|\, v(\eta) < v_{x,\lambda}(\eta)\}.$$

Lemma 3.3. *Assume the same conditions on n, α, θ, and κ as those in Theorem 3.1. Then for any $x \in \partial\mathbb{R}_+^n$, there exists $\lambda_0(x) > 0$ such that: $\forall\ 0 < \lambda < \lambda_0(x)$,*

$$
\begin{aligned}
u_{x,\lambda}(\xi) &\le u(\xi), \quad \textit{a.e. in } \Sigma_{x,\lambda}^{n-1}, \\
v_{x,\lambda}(\eta) &\le v(\eta), \quad \textit{a.e. in } \Sigma_{x,\lambda}^{n}.
\end{aligned}
$$

Proof. For $\xi \in \Sigma_{x,\lambda}^{u}$, we have, via (3.4) and mean value theorem, that

$$
\begin{aligned}
0 \le u_{x,\lambda}(\xi) - u(\xi) &= \int_{\Sigma_{x,\lambda}^{n}} P(x,\lambda;\xi,\eta)\big[v_{x,\lambda}^{\beta}(\eta) - v^{\beta}(\eta)\big]d\eta, \\
&\le \int_{\Sigma_{x,\lambda}^{v}} P(x,\lambda;\xi,\eta)\big[v_{x,\lambda}^{\beta}(\eta) - v^{\beta}(\eta)\big]d\eta, \\
&\le \int_{\Sigma_{x,\lambda}^{v}} \frac{v_{x,\lambda}^{\beta}(\eta) - v^{\beta}(\eta)}{|\xi-\eta|^{n-\alpha}}d\eta, \\
&= \beta\int_{\Sigma_{x,\lambda}^{v}} \frac{\phi_\lambda(v)^{\beta-1}(v_{x,\lambda}(\eta) - v(\eta))}{|\xi-\eta|^{n-\alpha}}d\eta, \\
&\le \beta\int_{\Sigma_{x,\lambda}^{v}} \frac{v_{x,\lambda}^{\beta-1}(\eta)(v_{x,\lambda}(\eta) - v(\eta))}{|\xi-\eta|^{n-\alpha}}d\eta,
\end{aligned}
$$

where $v(\eta) \le \phi_\lambda(v) \le v_{x,\lambda}(\eta)$ on $\Sigma_{x,\lambda}^{v}$.

For $t \in (1, \frac{n}{\alpha})$, take $k = \frac{(n-1)t}{n-\alpha t}, s = \frac{nt}{n-\alpha t}$. Using inequality (1.12), the above inequality yields

$$\|(u_{x,\lambda} - u)_+\|_{L^k(\partial\mathbb{R}_+^n)} \ \le \ c\|v_{x,\lambda}^{\beta-1}(v_{x,\lambda} - v)_+\|_{L^t(\mathbb{R}_+^n)},$$

where $f_+(x) = \max(f(x), 0)$. Note $\beta - 1 = 2\alpha/(n-\alpha)$. By Hölder inequality, we have

$$\|v_{x,\lambda}^{\beta-1}(v_{x,\lambda} - v)\|_{L^t(\Sigma_{x,\lambda}^v)} \leq \|v_{x,\lambda}\|^{\beta-1}_{L^{\frac{2n}{n-\alpha}}(\Sigma_{x,\lambda}^v)} \|v_{x,\lambda} - v\|_{L^s(\Sigma_{x,\lambda}^v)}.$$

Thus

$$\|u_{x,\lambda} - u\|_{L^k(\Sigma_{x,\lambda}^u)} \leq c\|v_{x,\lambda}\|^{\beta-1}_{L^{\frac{2n}{n-\alpha}}(\Sigma_{x,\lambda}^v)} \|v_{x,\lambda} - v\|_{L^s(\Sigma_{x,\lambda}^v)}. \tag{3.6}$$

On the other hand, for any $\eta \in \Sigma_{x,\lambda}^v$, we know from (3.5) that

$$v_{x,\lambda}(\eta) - v(\eta) \quad \leq \quad \gamma \int_{\Sigma_{x,\lambda}^u} \frac{u_{x,\lambda}^{\gamma-1}(\xi)(u_{x,\lambda}(\xi) - u(\xi))}{|\eta - \xi|^{n-\alpha}} d\xi,$$

Note for $1 < t < \frac{n}{\alpha}$, $s = \frac{nt}{n-\alpha t} > \frac{n}{n-\alpha}$ and $\frac{(n-1)s}{n+(\alpha-1)s} = \frac{(n-1)t}{n-t}$. Similarly, using HLS inequality (1.8) and Hölder inequality, we have

$$\begin{aligned} \|v_{x,\lambda} - v\|_{L^s(\Sigma_{x,\lambda}^v)} &\leq c\|u_{x,\lambda}^{\gamma-1}(u_{x,\lambda} - u)\|_{L^{\frac{(n-1)s}{n+(\alpha-1)s}}(\Sigma_{x,\lambda}^u)} \\ &\leq c\|u_{x,\lambda}\|^{\gamma-1}_{L^{\frac{2(n-1)}{n-\alpha}}(\Sigma_{x,\lambda}^u)} \|u_{x,\lambda} - u\|_{L^k(\Sigma_{x,\lambda}^u)}. \end{aligned} \tag{3.7}$$

Combining (3.6) with (3.7), we obtain

$$\begin{aligned} & \|u_{x,\lambda} - u\|_{L^k(\Sigma_{x,\lambda}^u)} \\ \leq\ & c\|v_{x,\lambda}\|^{\beta-1}_{L^{\frac{2n}{n-\alpha}}(\Sigma_{x,\lambda}^v)} \|u_{x,\lambda}\|^{\gamma-1}_{L^{\frac{2(n-1)}{n-\alpha}}(\Sigma_{x,\lambda}^u)} \|u_{x,\lambda} - u\|_{L^k(\Sigma_{x,\lambda}^u)} \\ \leq\ & c\|v_{x,\lambda}\|^{\beta-1}_{L^{\frac{2n}{n-\alpha}}(\Sigma_{x,\lambda}^n)} \|u_{x,\lambda}\|^{\gamma-1}_{L^{\frac{2(n-1)}{n-\alpha}}(\Sigma_{x,\lambda}^{n-1})} \|u_{x,\lambda} - u\|_{L^k(\Sigma_{x,\lambda}^u)} \\ =\ & c\|v\|^{\beta-1}_{L^{\frac{2n}{n-\alpha}}(B_\lambda^+(x))} \|u\|^{\gamma-1}_{L^{\frac{2(n-1)}{n-\alpha}}(B_\lambda^{n-1}(x))} \|u_{x,\lambda} - u\|_{L^k(\Sigma_{x,\lambda}^u)}. \end{aligned} \tag{3.8}$$

Similarly, we have

$$\|v_{x,\lambda} - v\|_{L^s(\Sigma_{x,\lambda}^v)} \quad \leq \quad c\|v\|^{\beta-1}_{L^{\frac{2n}{n-\alpha}}(B_\lambda^+(x))} \|u\|^{\gamma-1}_{L^{\frac{2(n-1)}{n-\alpha}}(B_\lambda^{n-1}(x))} \|v_{x,\lambda} - v\|_{L^s(\Sigma_{x,\lambda}^v)}. \tag{3.9}$$

Since $u \in L_{loc}^{\frac{2(n-1)}{n-\alpha}}(\partial\mathbb{R}_+^n)$ and $v \in L_{loc}^{\frac{2n}{n-\alpha}}(\mathbb{R}_+^n)$, we can choose λ_0 small enough such that for $0 < \lambda < \lambda_0$, we have

$$c\|v\|^{\beta-1}_{L^{\frac{2n}{n-\alpha}}(B_\lambda^+(x))} \|u\|^{\gamma-1}_{L^{\frac{2(n-1)}{n-\alpha}}(B_\lambda^{n-1}(x))} \leq \frac{1}{2}.$$

Combining the above with (3.8) and (3.9), we get

$$\begin{aligned} \|u_{x,\lambda} - u\|_{L^k(\Sigma_{x,\lambda}^u)} &\leq \frac{1}{2}\|u_{x,\lambda} - u\|_{L^k(\Sigma_{x,\lambda}^u)}, \\ \|v_{x,\lambda} - v\|_{L^s(\Sigma_{x,\lambda}^v)} &\leq \frac{1}{2}\|v_{x,\lambda} - v\|_{L^s(\Sigma_{x,\lambda}^v)}, \end{aligned}$$

which imply that $\|u_{x,\lambda} - u\|_{L^k(\Sigma_{x,\lambda}^u)} = \|v_{x,\lambda} - v\|_{L^s(\Sigma_{x,\lambda}^v)} = 0$. That is, the measures for both $\Sigma_{x,\lambda}^u$ and $\Sigma_{x,\lambda}^v$ are zero. We complete the proof of the lemma. □

Define, module sets with zero measure,

$$\bar{\lambda}(x) = sup\{\mu > 0 \,|\, u_{x,\lambda}(\xi) \leq u(\xi), \text{and } v_{x,\lambda}(\eta) \leq v(\eta),\ \forall \lambda \in (0, \mu), \forall \xi \in \Sigma_{x,\lambda}^{n-1}, \forall \eta \in \Sigma_{x,\lambda}^n\}.$$

We will show: if the sphere stops, then we have symmetric properties for solutions.

Lemma 3.4. *For some* $x_0 \in \partial\mathbb{R}^n_+$, *if* $\bar\lambda(x_0) < \infty$, *then*

$$\begin{aligned} u_{x_0,\bar\lambda(x_0)} &= u \quad \text{on } \partial\mathbb{R}^n_+, \\ v_{x_0,\bar\lambda(x_0)} &= v \quad \text{on } \mathbb{R}^n_+. \end{aligned}$$

Proof. Denote $\bar\lambda = \bar\lambda(x_0)$. We only need to show

$$\begin{aligned} u_{x_0,\bar\lambda}(\xi) &= u(\xi) \quad \text{for all } \xi \in \Sigma^{n-1}_{x_0,\bar\lambda}, \\ v_{x_0,\bar\lambda}(\eta) &= v(\eta) \quad \text{for all } \eta \in \Sigma^{n}_{x_0,\bar\lambda}. \end{aligned}$$

By the definition of $\bar\lambda$, we have

$$\begin{aligned} u_{x_0,\bar\lambda}(\xi) &\le u(\xi) \quad \text{for all } \xi \in \Sigma^{n-1}_{x_0,\bar\lambda}, \\ v_{x_0,\bar\lambda}(\eta) &\le v(\eta) \quad \text{for all } \eta \in \Sigma^{n}_{x_0,\bar\lambda}. \end{aligned}$$

If $u_{x_0,\bar\lambda}(\xi) \not\equiv u(\xi)$ or $v_{x_0,\bar\lambda}(\eta) \not\equiv v(\eta)$, we know from (3.4) and (3.5) (using $x_0, \bar\lambda$ to replace x, λ), that

$$\begin{aligned} u_{x_0,\bar\lambda}(\xi) &< u(\xi) \quad \text{for all } \xi \in \Sigma^{n-1}_{x_0,\bar\lambda}, \\ v_{x_0,\bar\lambda}(\eta) &< v(\eta) \quad \text{for all } \eta \in \Sigma^{n}_{x_0,\bar\lambda}. \end{aligned}$$

Thus, for a given large R, ε_0 and any $\delta > 0$ there exist c_1, c_2 such that

$$\begin{aligned} u(\xi) - u_{x_0,\bar\lambda}(\xi) &> c_1 \quad \text{for } \xi \in \Sigma^{n-1}_{x_0,\bar\lambda+\delta} \cap B^{n-1}_R(x_0), \\ v(\eta) - v_{x_0,\bar\lambda}(\eta) &> c_2 \quad \text{for } \eta \in \Sigma^{n}_{x_0,\bar\lambda+\delta} \cap B^{+}_R(x_0). \end{aligned}$$

By (3.4) and (3.5), we know that we can choose $\varepsilon < \delta$ sufficiently small so that for $\lambda \in [\bar\lambda, \bar\lambda + \varepsilon)$,

$$\begin{aligned} u(\xi) &\ge u_{x_0,\lambda}(\xi) \quad \text{for } \xi \in \Sigma^{n-1}_{x_0,\bar\lambda+\delta} \cap B^{n-1}_R(x_0), \\ v(\eta) &\ge v_{x_0,\lambda}(\eta) \quad \text{for } \eta \in \Sigma^{n}_{x_0,\bar\lambda+\delta} \cap B^{+}_R(x_0). \end{aligned}$$

These imply that $\Sigma^u_{x_0,\lambda}$ and $\Sigma^v_{x_0,\lambda}$ have no intersection with

$$\Sigma^{n-1}_{x_0,\bar\lambda+\delta} \cap B^{n-1}_R(x_0) \quad \text{and} \quad \Sigma^{n}_{x_0,\bar\lambda+\delta} \cap B^{+}_R(x_0),$$

respectively. Thus, $\Sigma^u_{x_0,\lambda}$ is contained in the union of

$$\partial\mathbb{R}^n_+ \setminus B^{n-1}_R(x_0) \quad \text{and} \quad \Sigma^{n-1}_{x_0,\lambda} \backslash \Sigma^{n-1}_{x_0,\bar\lambda+\delta};$$

And $\Sigma^v_{x_0,\lambda}$ is contained in the union of

$$\mathbb{R}^n_+ \setminus B^{+}_R(x_0) \quad \text{and } \Sigma^{n}_{x_0,\lambda} \backslash \Sigma^{n}_{x_0,\bar\lambda+\delta}.$$

For simplicity, we write $\Omega^{n-1}_{\lambda,R} = (\partial\mathbb{R}^n_+ \setminus B^{n-1}_R(x_0)) \cup (\Sigma^{n-1}_{x_0,\lambda} \backslash \Sigma^{n-1}_{x_0,\bar\lambda+\delta})$, $\Omega^n_{\lambda,R} = (\mathbb{R}^n_+ \setminus B^+_R(x_0)) \cup (\Sigma^n_{x_0,\lambda} \backslash \Sigma^n_{x_0,\bar\lambda+\delta})$. Moreover, denote $(\Omega^n_{\lambda,R})^*$ and $(\Omega^{n-1}_{\lambda,R})^*$ as the reflection of $\Omega^n_{\lambda,R}$ and $\Omega^{n-1}_{\lambda,R}$ under the Kelvin transformation with respect to the sphere $\{x : |x - x_0| = \lambda\}$, respectively. That is, $(\Omega^n_{\lambda,R})^* = B^+_{\varepsilon_1}(x_0) \cup (B^+_\lambda(x_0) \backslash B^+_{\lambda^2/(\bar\lambda+\delta)}(x_0))$, and $(\Omega^{n-1}_{\lambda,R})^* = B^{n-1}_{\varepsilon_1}(x_0) \cup (B^{n-1}_\lambda(x_0) \backslash B^{n-1}_{\lambda^2/(\bar\lambda+\delta)}(x_0))$, where $\varepsilon_1 = \lambda/R$ is small as $R \to \infty$.

Similar to (3.8) and (3.9), for $\lambda \in [\bar\lambda, \bar\lambda+\varepsilon)$, we have

$$\begin{aligned}
\|u_{x,\lambda}-u\|_{L^k(\Sigma^u_{x_0,\lambda})} &\le c\|v_{x,\lambda}\|^{\beta-1}_{L^{\frac{2n}{n-\alpha}}(\Sigma^v_{x_0,\lambda})}\|u_{x,\lambda}\|^{\gamma-1}_{L^{\frac{2(n-1)}{n-\alpha}}(\Sigma^u_{x_0,\lambda})} \\
&\quad \times\|u_{x,\lambda}-u\|_{L^k(\Sigma^u_{x_0,\lambda})}, \\
&\le c\|v\|^{\beta-1}_{L^{\frac{2n}{n-\alpha}}((\Omega^n_{\lambda,R})^*)}\|u\|^{\gamma-1}_{L^{\frac{2(n-1)}{n-\alpha}}((\Omega^{n-1}_{\lambda,R})^*)} \\
&\quad \times\|u_{x,\lambda}-u\|_{L^k(\Sigma^u_{x_0,\lambda})}, && (3.10)\\
\|v_{x,\lambda}-v\|_{L^s(\Sigma^v_{x_0,\lambda})} &\le c\|v\|^{\beta-1}_{L^{\frac{2n}{n-\alpha}}((\Omega^n_{\lambda,R})^*)}\|u\|^{\gamma-1}_{L^{\frac{2(n-1)}{n-\alpha}}((\Omega^{n-1}_{\lambda,R})^*)} \\
&\quad \times\|v_{x,\lambda}-v\|_{L^s(\Sigma^v_{x_0,\lambda})}, && (3.11)
\end{aligned}$$

Since $u \in L^{\frac{2(n-1)}{n-\alpha}}_{loc}(\partial\mathbb{R}^n_+)$ and $v \in L^{\frac{2n}{n-\alpha}}_{loc}(\mathbb{R}^n_+)$, we have

$$\int_{(\Omega^{n-1}_{\lambda,R})^*} u^{\frac{2(n-1)}{n-\alpha}}(\xi)d\xi < \varepsilon_0(\varepsilon,\delta) \qquad \int_{(\Omega^n_{\lambda,R})^*} v^{\frac{2n}{n-\alpha}}(\eta)d\eta < \varepsilon_0(\varepsilon,\delta).$$

Choose ε_0 small enough (via choosing ε, δ small enough) such that for $\lambda \in [\bar\lambda, \bar\lambda+\varepsilon)$,

$$c\|v\|^{\beta-1}_{L^{\frac{2n}{n-\alpha}}((\Omega^{n-1}_{\lambda,R})^*)}\|u\|^{\gamma-1}_{L^{\frac{2(n-1)}{n-\alpha}}((\Omega^{n-1}_{\lambda,R})^*)} < \frac{1}{2}.$$

Substituting the above into (3.10) and (3.11), we obtain

$$\|u_{x,\lambda}-u\|_{L^k(\Sigma^u_{x_0,\lambda})} = \|v_{x,\lambda}-v\|_{L^s(\Sigma^v_{x_0,\lambda})} = 0.$$

Thus we conclude that

$$\begin{aligned}
u_{x_0,\lambda}(\xi) &\le u(\xi) \quad \text{for all } \xi \in \Sigma^{n-1}_{x_0,\lambda}, \\
v_{x_0,\lambda}(\eta) &\le v(\eta) \quad \text{for all } \eta \in \Sigma^n_{x_0,\lambda}
\end{aligned}$$

for $\lambda \in [\bar\lambda, \bar\lambda+\varepsilon)$, which contradicts the definition of $\bar\lambda$. □

The following three calculus key lemmas are needed for carrying out moving sphere procedure. Under stronger assumptions ($f \in C^1(\mathbb{R}^n_+)$), these lemmas were early proved by Li and Zhu [36], and Li and Zhang [35]. The first two lemmas, due to Li and Nirenberg, are adopted from Li [34].

Lemma 3.5. *(Lemma 5.7 in [34]) For $n \ge 1$ and $\mu \in \mathbb{R}$, if f is a function defined on $\mathbb{R}^n$ and valued in $(-\infty, +\infty)$ satisfying*

$$\big(\frac{\lambda}{|y-x|}\big)^\mu f\big(x+\frac{\lambda^2(y-x)}{|y-x|^2}\big) \le f(y), \quad \forall\lambda>0,\ |y-x|\ge\lambda, x,y\in\mathbb{R}^n,$$

then $f(x) = constant$.

Lemma 3.6. *(Lemma 5.8 in [34]) Let $n \ge 1$ and $\mu \in \mathbb{R}$, and $f \in C^0(\mathbb{R}^n)$. Suppose that for every $x \in \mathbb{R}^n$, there exists $\lambda > 0$ such that*

$$\big(\frac{\lambda}{|y-x|}\big)^\mu f\big(x+\frac{\lambda^2(y-x)}{|y-x|^2}\big) = f(y), \quad \forall y\in\mathbb{R}^n\setminus\{x\}.$$

Then there are $a \ge 0, d > 0$ and $\bar x \in \mathbb{R}^n$, such that

$$f(x) \equiv \pm a\big(\frac{1}{d+|x-\bar x|^2}\big)^{\frac{\mu}{2}}.$$

Lemma 3.7. *For $n \geq 1$ and $\mu \in \mathbb{R}$, if f is a function defined on $\mathbb{R}^n_+$ and valued in $(-\infty, +\infty)$ satisfying*

$$\big(\frac{\lambda}{|y-x|}\big)^{\mu} f\big(x + \frac{\lambda^2(y-x)}{|y-x|^2}\big) \leq f(y), \quad \forall \lambda > 0,\ |y-x| \geq \lambda, y \in \mathbb{R}^n_+, x \in \partial\mathbb{R}^n_+,$$

then

$$f(z) = f(z', t) = f(0, t),\ \forall\, z = (z', t) \in \mathbb{R}^n_+.$$

Proof. The proof of this lemma is similar to that of Lemma 7.1 in [34]. For any $z = (z', z_n) \in \mathbb{R}^n_+$, choose $y^i = (y', y^i_n)$ with $y' \neq z'$, $y^i_n > z_n$ and $y^i_n \to z_n$ as $i \to \infty$. Choose $b^i > 1$, so that

$$x^i := (x'^i, 0) = x^i(b^i) = y^i + b^i(z - y^i) \in \partial\mathbb{R}^n_+.$$

Also define

$$\lambda_i := \lambda_i(b^i) = \sqrt{|z - x^i||y^i - x^i|}.$$

Then,

$$z = x^i + \frac{\lambda_i^2(y^i - x^i)}{|y^i - x^i|^2},$$

and by the assumption of Lemma 3.7,

$$\big(\frac{\lambda_i}{|y^i - x^i|}\big)^{\mu} f(z) \leq f(y^i).$$

Since

$$\lim_{i\to\infty} \frac{\lambda_i}{|y^i - x^i|} = \lim_{i\to\infty} \sqrt{\frac{|z - x^i|}{|y^i - x^i|}} = 1, \quad \text{and} \quad \lim_{i\to\infty} y^i = (y', z_n),$$

we obtain $f(z', z_n) \leq f(y', z_n)$. Since y' and z' are arbitrary, we have the lemma.

□

Proof of Theorem 3.1.

Case 1. If there exists some $x_0 \in \partial\mathbb{R}^n_+$ such that $\bar\lambda(x_0) < \infty$, then $\bar\lambda(x) < \infty$ for all $x \in \partial\mathbb{R}^n_+$.

For any $x \in \partial\mathbb{R}^n_+$, from the definition of $\bar\lambda(x)$, we know $\forall\, \lambda \in (0, \bar\lambda(x))$,

$$u_{x,\lambda}(\xi) \leq u(\xi), \qquad \forall \xi \in \Sigma^{n-1}_{x,\lambda}.$$

It implies

$$a := \liminf_{|\xi|\to\infty} \big(|\xi|^{n-\alpha} u(\xi)\big) \geq \liminf_{|\xi|\to\infty} \big(|\xi|^{n-\alpha} u_{x,\lambda}(\xi)\big) = \lambda^{n-\alpha} u(x),\ \forall\, \lambda \in (0, \bar\lambda(x)). \tag{3.12}$$

On the other hand, since $\bar\lambda(x_0) < \infty$, using Lemma 3.4 we have

$$a = \liminf_{|\xi|\to\infty} \big(|\xi|^{n-\alpha} u(\xi)\big) = \liminf_{|\xi|\to\infty} \big(|\xi|^{n-\alpha} u_{x_0,\bar\lambda}(\xi)\big) = \bar\lambda^{n-\alpha} u(x_0) < \infty. \tag{3.13}$$

Combining (3.12) with (3.13) we obtain $\bar\lambda(x) < \infty$ for all $x \in \partial\mathbb{R}^n_+$. Applying Lemma 3.4 again, we know

$$u_{x,\bar\lambda}(\xi) = u(\xi),\ \forall\, x, \xi \in \partial\mathbb{R}^n_+.$$

By Lemma 3.6, we have: for all $x \in \partial\mathbb{R}^n_+$,

$$u(\xi) = c_1\big(\frac{1}{|\xi - \xi_0|^2 + d^2}\big)^{\frac{n-\alpha}{2}}$$

for some c_1, $d > 0$ and $\xi_0 \in \partial\mathbb{R}^n_+$.

Bringing the above into the second equation in system (3.1), we can show, for $\xi \in \partial\mathbb{R}^n_+$, that

$$v(\xi,0) = c_2\Big(\frac{1}{|\xi-\xi_0|^2+d^2}\Big)^{\frac{n-\alpha}{2}}$$

for some c_2, $d>0$ and $\xi_0 \in \partial\mathbb{R}^n_+$. More computational details can be found in the proof of Lemma 6.1 in Li [34].

Case 2. $\bar{\lambda}(x)=\infty$ for all $x\in\partial\mathbb{R}^n_+$.

Then for any given $x \in \partial\mathbb{R}^n_+$,

$$u_{x,\lambda}(\xi) \le u(\xi),\ \forall\, \xi \in \Sigma^{n-1}_{x,\lambda}.$$

Using Lemma 3.5, we conclude that $u = C_0$ is a constant.

On the other hand, for any $x\in\partial\mathbb{R}^n_+$,

$$v_{x,\lambda}(\eta) \le v(\eta),\ \forall\, \eta \in \Sigma^{n}_{x,\lambda}.$$

From Lemma 3.7, we conclude that v only depends on t. Thus, we have

$$v(\eta',t) = v(0,t) = \int_{\partial\mathbb{R}^n_+} \frac{C_0^\gamma}{(|y|^2+t^2)^{\frac{n-\alpha}{2}}}dy,$$

However,

$$\begin{aligned} v(0,t) &= \int_{\partial\mathbb{R}^n_+} \frac{C_0^\gamma}{(|y|^2+t^2)^{\frac{n-\alpha}{2}}}dy \\ &= Ct^{\alpha-1}\int_0^\infty \frac{\rho^{n-2}}{(\rho^2+1)^{\frac{n-\alpha}{2}}}d\rho. \end{aligned}$$

Since $\alpha>1$, we conclude that $v(0,t)$ is infinite for $t\ne 0$. Contradiction. □

Remark 3.8. It is interesting to point out that Theorem 3.1 can also be directly proved from the fact that $\bar{\lambda}(x)<\infty$ for all $x\in\partial\mathbb{R}^n_+$ via Theorem 1.4 in [20] without using the C^0 regularity of solutions.

4. **Regularity of solutions to integral equation**

In this section, we address the regularity properties of solutions to integral equation (1.13).

Theorem 4.1. *Let $1<\alpha<n$ and $1<p<(n-1)/(\alpha-1)$. Suppose that $f\in L^p_{loc}(\partial\mathbb{R}^n_+)$ is nonnegative solution to* (1.13) *with $\frac{1}{q}=\frac{n-1}{n}\big(\frac{1}{p}-\frac{\alpha-1}{n-1}\big)$. Then $f\in C^\infty(\partial\mathbb{R}^n_+)$.*

Theorem 4.1 is equivalent to the following.

Theorem 4.2. *Assume $1<\alpha<n$, $\frac{\alpha-1}{n-\alpha}<\theta<\infty$ and $0<\kappa<\infty$ given by*

$$\frac{1}{\kappa+1} = \frac{n-1}{n}\Big(\frac{n-\alpha}{n-1}-\frac{1}{\theta+1}\Big). \tag{4.1}$$

If (u,v) is a pair of positive solutions of (3.1) *with $u\in L^{\theta+1}_{loc}(\partial\mathbb{R}^n_+)$, then $u\in C^\infty(\partial\mathbb{R}^n_+)$ and $v\in C^\infty(\overline{\mathbb{R}^n_+})$.*

To prove this theorem, we first establish two local regularity results, which are similar to Lemma A.1 in [6], Theorem 1.3 in [34] and Proposition 5.2 and 5.3 in [27], and quite similar in spirit to Brezis and Kato's Lemma [6] and the regularity result in [4].

We use the notations from previous section. And later in this section, for $x=0$, we write $B_R = B_R(0), B_R^{n-1} = B_R^{n-1}(0), B_R^+ = B_R^+(0), \Sigma_R^n = \Sigma_{0,R}^n$ and $\Sigma_R^{n-1} = \Sigma_{0,R}^{n-1}$.

Proposition 4.3. *Assume $1<\alpha<n$, $1<a,b\le\infty, 1\le r<\infty$ and $\frac{n}{n-\alpha}<p<q<\infty$ satisfy*

$$\frac{\alpha}{n} < \frac{r}{q} + \frac{1}{a} < \frac{r}{p} + \frac{1}{a} < 1 \quad \text{and} \quad \frac{n}{ar} + \frac{n-1}{b} = \frac{\alpha}{r} + (\alpha - 1). \tag{4.2}$$

Suppose that $v, h \in L^p(B_R^+), V \in L^a(B_R^+)$, and $U \in L^b(B_R^{n-1})$ are all nonnegative functions with $h|_{B_{R/2}^+} \in L^q(B_{R/2}^+)$, and

$$v(x) \le \int_{B_R^{n-1}} \frac{U(y)}{|x-y|^{n-\alpha}} \big[\int_{B_R^+} \frac{V(z) v^r(z)}{|z-y|^{n-\alpha}} dz \big]^{\frac{1}{r}} dy + h(x), \qquad \forall x \in B_R^+.$$

There is a $\varepsilon = \varepsilon(n,\alpha,p,q,r,a,b) > 0$, and $C(n,\alpha,p,q,a,b,r,\varepsilon) > 0$ such that if

$$\|U\|_{L^b(B_R^{n-1})} \|V\|_{L^a(B_R^+)}^{\frac{1}{r}} \le \varepsilon(n,\alpha,p,q,r,a,b),$$

then

$$\|v\|_{L^q(B_{R/4}^+)} \le C(n,\alpha,p,q,a,b,r,\varepsilon)\big(R^{\frac{n}{q}-\frac{n}{p}} \|v\|_{L^p(B_R^+)} + \|h\|_{L^q(B_{R/2}^+)}\big).$$

Proof. After rescaling, we may assume $R=1$.

We first consider the case that $v,\ h \in L^q(B_1^+)$. Denote

$$u(y) = \int_{B_1^+} \frac{V(x) v^r(x)}{|x-y|^{n-\alpha}} dx \quad \text{for} \quad y \in B_1^{n-1},$$

and define

$$\psi(y) = \begin{cases} u(y) & y \in B_1^{n-1}, \\ 0 & y \notin B_1^{n-1}, \end{cases}$$

and

$$\phi(x) = \begin{cases} v(x) & x \in B_1^+, \\ 0 & x \notin B_1^+. \end{cases}$$

For any $s_1, s_2 \in (1, n/\alpha)$, we know from inequality (1.12), that

$$\|\psi\|_{L^{p_1}(\partial \mathbb{R}_+^n)} \le c(n,\alpha,s_1,r) \|V\phi^r\|_{L^{s_1}(\mathbb{R}_+^n)},$$
$$\|\psi\|_{L^{q_1}(\partial \mathbb{R}_+^n)} \le c(n,\alpha,s_2,r) \|V\phi^r\|_{L^{s_2}(\mathbb{R}_+^n)},$$

where

$$\frac{1}{p_1} = \frac{n}{n-1}\big(\frac{1}{s_1} - \frac{\alpha}{n}\big), \quad \frac{1}{q_1} = \frac{n}{n-1}\big(\frac{1}{s_2} - \frac{\alpha}{n}\big).$$

In particular, if we choose s_1, s_2 so that $\frac{1}{s_1} = \frac{r}{p} + \frac{1}{a}, \frac{1}{s_2} = \frac{r}{q} + \frac{1}{a}$, we have, via Hölder inequality, that

$$\|u\|_{L^{p_1}(B_1^{n-1})} \le c(n,\alpha,p,a,r) \|V\|_{L^a(B_1^+)} \|v\|_{L^p(B_1^+)}^r, \tag{4.3}$$
$$\|u\|_{L^{q_1}(B_1^{n-1})} \le c(n,\alpha,q,a,r) \|V\|_{L^a(B_1^+)} \|v\|_{L^q(B_1^+)}^r, \tag{4.4}$$

where p_1 and q_1 satisfy

$$\frac{1}{p_1} = \frac{n}{n-1}\big(\frac{r}{p} + \frac{1}{a} - \frac{\alpha}{n}\big), \quad \frac{1}{q_1} = \frac{n}{n-1}\big(\frac{r}{q} + \frac{1}{a} - \frac{\alpha}{n}\big).$$

The existence of p_1, q_1 is guaranteed by (4.2). Let $0<\varrho<\delta\le\frac{1}{2}$. For $x\in B_\delta^+$, we have

$$\begin{aligned} v(x) &\le \int_{B^{n-1}_{\frac{\delta+\varrho}{2}}}\frac{U(y)u^{\frac{1}{r}}(y)}{|x-y|^{n-\alpha}}dy+\int_{B_1^{n-1}\setminus B^{n-1}_{\frac{\delta+\varrho}{2}}}\frac{U(y)u^{\frac{1}{r}}(y)}{|x-y|^{n-\alpha}}dy+h(x)\\ &=: J_1(x)+J_2(x)+h(x). \end{aligned}$$

From (4.2), we know $\frac{1}{q}=\frac{n-1}{n}\big(\frac{1}{b}+\frac{1}{q_1 r}-\frac{\alpha-1}{n-1}\big)$. Using inequality (1.8) and Hölder inequality, we have

$$\|J_1\|_{L^q(B_\varrho^+)}\le c(n,\alpha,q,r,b)\|U\|_{L^b(B_1^{n-1})}\|u\|^{\frac{1}{r}}_{L^{q_1}(B^{n-1}_{\frac{\delta+\varrho}{2}})}.$$

Let $\frac{1}{m_1}=1-\frac{1}{b}-\frac{1}{p_1 r}$. Since $p>n/(n-\alpha)$, we know $m_1>1$. From Hölder inequality and (4.3), we know

$$\begin{aligned} J_2(x) &\le \frac{c(n,\alpha)}{(\delta-\varrho)^{n-\alpha}}\int_{B_1^{n-1}\setminus B^{n-1}_{\frac{\delta+\varrho}{2}}}U(y)u^{\frac{1}{r}}(y)dy\\ &\le \frac{c(n,\alpha,p,r,b)}{(\delta-\varrho)^{n-\alpha}}|B_1^{n-1}|^{\frac{1}{m_1}}\|U\|_{L^b(B_1^{n-1})}\|u\|^{\frac{1}{r}}_{L^{p_1}(B_1^{n-1})}\\ &\le \frac{c(n,\alpha,p,r,a,b)}{(\delta-\varrho)^{n-\alpha}}|B_1^{n-1}|^{\frac{1}{m_1}}\|U\|_{L^b(B_1^{n-1})}\|V\|^{\frac{1}{r}}_{L^a(B_1^+)}\|v\|_{L^p(B_1^+)}. \end{aligned}$$

Combining the above and using Minkowski inequality we have

$$\begin{aligned} \|v\|_{L^q(B_\varrho^+)} \le\ & c(n,\alpha,q,r,b)\|U\|_{L^b(B_1^{n-1})}\|u\|^{\frac{1}{r}}_{L^{q_1}(B^{n-1}_{\frac{\delta+\varrho}{2}})}+\frac{c(n,\alpha,p,q,r,a,b)}{(\delta-\varrho)^{n-\alpha}}|B_1^{n-1}|^{\frac{1}{m_1}}|B_\varrho^+|^{\frac{1}{q}}\\ &\times\|U\|_{L^b(B_1^{n-1})}\|V\|^{\frac{1}{r}}_{L^a(B_1^+)}\|v\|_{L^p(B_1^+)}+\|h\|_{L^q(B^+_{\frac{1}{2}})}. \end{aligned} \tag{4.5}$$

On the other hand, for $y\in B^{n-1}_{\frac{\delta+\varrho}{2}}$, we have

$$\begin{aligned} u(y) &= \int_{B_\delta^+}\frac{V(x)v^r(x)}{|x-y|^{n-\alpha}}dx+\int_{B_1^+\setminus B_\delta^{n-1}}\frac{V(x)v^r(x)}{|x-y|^{n-\alpha}}dx\\ &=: K_1(y)+K_2(y). \end{aligned}$$

Using (4.4) on $B^{n-1}_{\frac{\delta+\varrho}{2}}$, we have

$$\|K_1\|_{L^{q_1}(B^{n-1}_{\frac{\delta+\varrho}{2}})}\le c(n,\alpha,q,r,a)\|V\|_{L^a(B_1^+)}\|v\|^r_{L^q(B_\delta^+)}.$$

For $\frac{1}{m_2}=1-\frac{1}{a}-\frac{r}{p}$,

$$\begin{aligned} K_2(y) &\le \frac{c(n,\alpha)}{(\delta-\varrho)^{n-\alpha}}\int_{B_1^+\setminus B_\delta^+}V(x)v^r(x)dx\\ &\le \frac{c(n,\alpha,p,a,r)}{(\delta-\varrho)^{n-\alpha}}|B_1^+|^{\frac{1}{m_2}}\|V\|_{L^a(B_1^+)}\|v\|^r_{L^p(B_1^+)}. \end{aligned}$$

Combining the above and using Minkowski inequality we have

$$\begin{aligned}\|u\|_{L^{q_1}(B^{n-1}_{\frac{\delta+\varrho}{2}})} &\le c(n,\alpha,q,r,a)\|V\|_{L^a(B_1^+)}\|v\|^r_{L^q(B_\delta^+)} + \frac{c(n,\alpha,p,q,r,a,b)}{(\delta-\varrho)^{n-\alpha}} \\ &\quad \times |B_1^+|^{\frac{1}{m_2}}|B_\delta^{n-1}|^{\frac{1}{q_1}}\|V\|_{L^a(B_1^+)}\|v\|^r_{L^p(B_1^+)}. \qquad (4.6)\end{aligned}$$

Bringing (4.6) into (4.5) and choosing ε small enough in condition

$$\|U\|_{L^b(B_1^{n-1})}\|V\|^{\frac{1}{r}}_{L^a(B_1^+)} \le \varepsilon(n,\alpha,p,q,r,a,b),$$

we have

$$\begin{aligned}\|v\|_{L^q(B_\varrho^+)} &\le \frac{1}{2}\|v\|_{L^q(B_\delta^+)} + c(n,\alpha,p,q,r,a,b,\varepsilon) \\ &\quad \times \Big(\frac{|B_1^{n-1}|^{\frac{1}{m_1}}|B_1^+|^{\frac{1}{q}}}{(\delta-\varrho)^{n-\alpha}} + \frac{|B_1^+|^{\frac{1}{rm_2}}|B_1^{n-1}|^{\frac{1}{rq_1}}}{(\delta-\varrho)^{\frac{n-\alpha}{r}}}\Big)\|v\|_{L^p(B_1^+)} + \|h\|_{L^q(B^+_{\frac{1}{2}})}.\end{aligned}$$

Then the usual iteration procedure (see, e.g. Lemma 4.1 in Chen and Wu [13], P_{27}) yields

$$\|v\|_{L^q(B^+_{\frac{1}{4}})} \le c(n,\alpha,p,q,r,a,b,\varepsilon)(\|v\|_{L^p(B_1^+)} + \|h\|_{L^q(B^+_{\frac{1}{2}})}).$$

For general $v,\ h \in L^p(B_1^+)$, we follow the argument given in [10]. See, also, [27]. Let $0 \le \zeta(x) \le 1$ be the measurable function such that

$$v(x) = \zeta(x)\int_{B_1^{n-1}} \frac{U(y)}{|x-y|^{n-\alpha}}\Big[\int_{B_1^+}\frac{V(z)v^r(z)}{|z-y|^{n-\alpha}}dz\Big]^{\frac{1}{r}}dy + \zeta(x)h(x) \quad \text{for any} \quad x\in\mathbb{R}^n_+.$$

Define the map T by

$$T(\varphi)(x) = \zeta(x)\int_{B_1^{n-1}} \frac{U(y)}{|x-y|^{n-\alpha}}\Big[\int_{B_1^+}\frac{V(z)|\varphi(z)|^r}{|z-y|^{n-\alpha}}dz\Big]^{\frac{1}{r}}dy.$$

Similar to the above estimates, using inequality (1.8) and Hölder inequality, we have

$$\begin{aligned}\|T(\varphi)\|_{L^p(B_1^+)} &\le c(n,\alpha,p,r,a,b)\|U\|_{L^b(B_1^{n-1})}\|V\|^{\frac{1}{r}}_{L^a(B_1^+)}\|\varphi\|_{L^p(B_1^+)} \le \frac{1}{2}\|\varphi\|_{L^p(B_1^+)}, \\ \|T(\varphi)\|_{L^q(B_1^+)} &\le c(n,\alpha,q,r,a,b)\|U\|_{L^b(B_1^{n-1})}\|V\|^{\frac{1}{r}}_{L^a(B_1^+)}\|\varphi\|_{L^q(B_1^+)} \le \frac{1}{2}\|\varphi\|_{L^q(B_1^+)},\end{aligned}$$

for ε small enough. Furthermore, for $\varphi,\psi \in L^p(B_1^+)$, it follows from Minkowski inequality that

$$|T(\varphi)(x) - T(\psi)(x)| \le T(|\varphi-\psi|)(x), \quad \text{for } x\in B_1^+.$$

Hence,

$$\|T(\varphi)-T(\psi)\|_{L^p(B_1^+)} \le \|T(|\varphi-\psi|)\|_{L^p(B_1^+)} \le \frac{1}{2}\|\varphi-\psi\|_{L^p(B_1^+)}.$$

Similarly, for $\varphi,\psi\in L^q(B_1^+)$, we have

$$\|T(\varphi)-T(\psi)\|_{L^q(B_1^+)} \le \frac{1}{2}\|\varphi-\psi\|_{L^q(B_1^+)}.$$

Define $h_j(x) = \min\{h(x), j\}$. Then we conclude from the contraction mapping theorem that we may find a unique $v_j \in L^q(B_1^+)$ such that

$$\begin{aligned} v_j(x) &= T(v_j)(x) + \zeta(x)h_j(x) \\ &= \zeta(x)\int_{B_1^{n-1}} \frac{U(y)}{|x-y|^{n-\alpha}}\Big[\int_{B_1^+} \frac{V(z)v_j^r(z)}{|z-y|^{n-\alpha}}dz\Big]^{\frac{1}{r}}dy + \zeta(x)h_j(x). \end{aligned}$$

Using the *a priori* estimate for v_j (noting that $h_j \in L^q(B_1^+)$), we have

$$\|v_j\|_{L^q(B^+_{\frac{1}{4}})} \leq c(n,\alpha,p,q,r,a,b,\varepsilon)\big(\|v_j\|_{L^p(B_1^+)} + \|h_j\|_{L^q(B^+_{\frac{1}{2}})}\big). \tag{4.7}$$

Observe that

$$v(x) = T(v)(x) + \zeta(x)h(x).$$

We have

$$\begin{aligned} \|v_j - v\|_{L^p(B_1^+)} &\leq \|T(v_j) - T(v)\|_{L^p(B_1^+)} + \|h_j - h\|_{L^p(B_1^+)} \\ &\leq \frac{1}{2}\|v_j - v\|_{L^p(B_1^+)} + \|h_j - h\|_{L^p(B_1^+)}. \end{aligned}$$

This implies

$$\|v_j - v\|_{L^p(B_1^+)} \leq 2\|h_j - h\|_{L^p(B_1^+)} \to 0,$$

as $j \to \infty$. Note $h_j \to h$ in $L^q(B^+_{1/2})$. Sending j to ∞ in (4.7), we obtain Proposition 4.3. □

The dual local regularity result is the following.

Proposition 4.4. *Assume* $1 < \alpha < n$, $1 < a, b \leq \infty, 1 \leq r < \infty$ *and* $\frac{n-1}{n-\alpha} < p < q < \infty$ *satisfy*

$$\frac{\alpha-1}{n-1} < \frac{r}{q} + \frac{1}{a} < \frac{r}{p} + \frac{1}{a} < 1 \quad \text{and} \quad \frac{n-1}{ar} + \frac{n}{b} = \frac{\alpha-1}{r} + \alpha. \tag{4.8}$$

Suppose $u, g \in L^p(B_R^{n-1}), U \in L^a(B_R^{n-1}), V \in L^b(B_R^+)$ *are all nonnegative functions with* $g|_{B^{n-1}_{R/2}} \in L^q(B^{n-1}_{R/2})$, *and*

$$u(y) \leq \int_{B_R^+} \frac{V(x)}{|x-y|^{n-\alpha}}\Big[\int_{B_R^{n-1}} \frac{U(z)u^r(z)}{|z-y|^{n-\alpha}}dz\Big]^{\frac{1}{r}}dx + g(y), \quad \forall y \in B_R^{n-1}.$$

There is a $\varepsilon = \varepsilon(n,\alpha,p,q,r,a,b) > 0$ *and* $C(n,\alpha,p,q,a,b,r,\varepsilon) > 0$, *such that if*

$$\|U\|^{\frac{1}{r}}_{L^a(B_R^{n-1})}\|V\|_{L^b(B_R^+)} \leq \varepsilon(n,\alpha,p,q,r,a,b),$$

then

$$\|u\|_{L^q(B^{n-1}_{R/4})} \leq C(n,\alpha,p,q,a,b,r,\varepsilon)\big(R^{\frac{n-1}{q}-\frac{n-1}{p}}\|u\|_{L^p(B_R^{n-1})} + \|g\|_{L^q(B^{n-1}_{R/2})}\big).$$

Proof. After rescaling, we may assume $R = 1$. We may further assume $u, g \in L^q(B_1^{n-1})$, since similar argument to that in the proof of Proposition 4.3 yields the same estimate under the assumption of $u, g \in L^p(B_1^{n-1})$.

Denote

$$v(x) = \int_{B_1^{n-1}} \frac{U(y)u^r(y)}{|x-y|^{n-\alpha}}dy \quad \text{for} \quad x \in B_1^+.$$

Let p_1 and q_1 be the numbers given by

$$\frac{1}{p_1} = \frac{n-1}{n}\big(\frac{r}{p} + \frac{1}{a} - \frac{\alpha-1}{n-1}\big), \quad \frac{1}{q_1} = \frac{n-1}{n}\big(\frac{r}{q} + \frac{1}{a} - \frac{\alpha-1}{n-1}\big).$$

Condition (4.8) indicates that $p_1,\, q_1 > 1$. Similar to Proposition 4.3, using (1.8), we have

$$\begin{aligned}
\|v\|_{L^{p_1}(B_1^+)} &\leq c(n,\alpha,p,a,r)\|U\|_{L^a(B_1^{n-1})}\|u\|^r_{L^p(B_1^{n-1})}, && (4.9)\\
\|v\|_{L^{q_1}(B_1^+)} &\leq c(n,\alpha,q,a,r)\|U\|_{L^a(B_1^{n-1})}\|u\|^r_{L^q(B_1^{n-1})}. && (4.10)
\end{aligned}$$

Let $0<\varrho<\delta\leq\frac{1}{2}$. For $y\in B_\delta^{n-1}$,

$$\begin{aligned}
u(x) &\leq \int_{B^+_{\frac{\delta+\varrho}{2}}}\frac{V(x)v^{\frac{1}{r}}(x)}{|x-y|^{n-\alpha}}dx+\int_{B_1^+\setminus B^+_{\frac{\delta+\varrho}{2}}}\frac{V(x)v^{\frac{1}{r}}(x)}{|x-y|^{n-\alpha}}dx+g(y)\\
&=: J_3(y)+J_4(y)+g(y).
\end{aligned}$$

From (4.8), we know $\frac{1}{q}=\frac{n}{n-1}\big(\frac{1}{b}+\frac{1}{q_1 r}-\frac{\alpha}{n}\big)$. Using inequality (1.12), we have

$$\|J_3\|_{L^q(B_\varrho^{n-1})}\leq c(n,\alpha,q,r,b)\|V\|_{L^b(B_1^+)}\|v\|^{\frac{1}{r}}_{L^{q_1}(B^+_{\frac{\delta+\varrho}{2}})}.$$

Let $\frac{1}{m_3}=1-\frac{1}{b}-\frac{1}{p_1 r}$ (note: $p>(n-1)/(n-\alpha)$ implies $m_3>1$). From Hölder inequality and (4.9), it follows

$$\begin{aligned}
J_4(y) &\leq \frac{c(n,\alpha)}{(\delta-\varrho)^{n-\alpha}}\int_{B_1^+\setminus B^+_{\frac{\delta+\varrho}{2}}}V(x)v^{\frac{1}{r}}(x)dx\\
&\leq \frac{c(n,\alpha,p,b,r)}{(\delta-\varrho)^{n-\alpha}}|B_1^+|^{\frac{1}{m_3}}\|V\|_{L^b(B_1^+)}\|v\|^{\frac{1}{r}}_{L^{p_1}(B_1^+)}\\
&\leq \frac{c(n,\alpha,p,q,r,a,b)}{(\delta-\varrho)^{n-\alpha}}|B_1^+|^{\frac{1}{m_3}}\|V\|_{L^b(B_1^+)}\|U\|^{\frac{1}{r}}_{L^a(B_1^{n-1})}\|u\|_{L^q(B_1^{n-1})}.
\end{aligned}$$

Combining the above and using Minkowski inequality, we have

$$\begin{aligned}
\|u\|_{L^q(B_\varrho^{n-1})} &\leq c(n,\alpha,q,r,b)\|V\|_{L^b(B_1^+)}\|v\|^{\frac{1}{r}}_{L^{q_1}(B^+_{\frac{\delta+\varrho}{2}})}+\frac{c(n,\alpha,p,q,r,a,b)}{(\delta-\varrho)^{n-\alpha}}|B_1^+|^{\frac{1}{m_3}}|B_\varrho^{n-1}|^{\frac{1}{q}}\\
&\quad\times\|V\|_{L^b(B_1^+)}\|U\|^{\frac{1}{r}}_{L^a(B_1^{n-1})}\|u\|_{L^q(B_1^{n-1})}+\|g\|_{L^q(B^{n-1}_{\frac{1}{2}})}. && (4.11)
\end{aligned}$$

On the other hand, for $x\in B^+_{\frac{\delta+\varrho}{2}}$, we have

$$\begin{aligned}
v(x) &= \int_{B_\delta^{n-1}}\frac{U(y)u^r(y)}{|x-y|^{n-\alpha}}dy+\int_{B_1^{n-1}\setminus B_\delta^{n-1}}\frac{U(y)u^r(y)}{|x-y|^{n-\alpha}}dy\\
&=: K_3(x)+K_4(x).
\end{aligned}$$

Using (4.10) on $B^+_{\frac{\delta+\varrho}{2}}$, we have

$$\|K_3\|_{L^{q_1}(B^+_{\frac{\delta+\varrho}{2}})}\leq c(n,\alpha,q,r,a)\|U\|_{L^a(B_1^{n-1})}\|u\|^r_{L^q(B_\delta^{n-1})}.$$

For $\frac{1}{m_4}=1-\frac{1}{a}-\frac{r}{p}$, from Hölder inequality, it yields

$$\begin{aligned}
K_4(x) &\leq \frac{c(n,\alpha)}{(\delta-\varrho)^{n-\alpha}}\int_{B_1^{n-1}\setminus B_\delta^{n-1}}U(y)u^r(y)dy\\
&\leq \frac{c(n,\alpha,p,a,r)}{(\delta-\varrho)^{n-\alpha}}|B_1^{n-1}|^{\frac{1}{m_4}}\|U\|_{L^a(B_1^{n-1})}\|u\|^r_{L^p(B_1^{n-1})}.
\end{aligned}$$

Combining the above and using Minkowski inequality, we have

$$\begin{aligned} \|v\|_{L^{q_1}(B^+_{\frac{\delta+\varrho}{2}})} &\leq c(n,\alpha,q,r,a)\|U\|_{L^a(B_1^{n-1})}\|u\|^r_{L^q(B_\delta^{n-1})} + \frac{c(n,\alpha,p,q,r,a,b)}{(\delta-\varrho)^{n-\alpha}} \\ &\times |B_1^{n-1}|^{\frac{1}{m_4}}|B_\delta^+|^{\frac{1}{q_1}}\|U\|_{L^a(B_1^{n-1})}\|u\|^r_{L^p(B_1^{n-1})}. \end{aligned} \tag{4.12}$$

Bringing (4.12) into (4.11), for ε small enough in

$$\|V\|_{L^b(B_1^+)}\|U\|^{\frac{1}{r}}_{L^a(B_1^{n-1})} \leq \varepsilon(n,\alpha,p,q,r,a,b)$$

we have

$$\begin{aligned} \|u\|_{L^q(B_\varrho^{n-1})} &\leq \frac{1}{2}\|u\|_{L^q(B_\delta^{n-1})} + c(n,\alpha,p,q,r,a,b,\varepsilon) \\ &\times \left(\frac{|B_1^+|^{\frac{1}{m_3}}|B_1^{n-1}|^{\frac{1}{q}}}{(\delta-\varrho)^{n-\alpha}} + \frac{|B_1^{n-1}|^{\frac{1}{rm_4}}|B_1^+|^{\frac{1}{rq_1}}}{(\delta-\varrho)^{\frac{n-\alpha}{r}}}\right)\|u\|_{L^p(B_1^{n-1})} + \|g\|_{L^q(B^{n-1}_{\frac{1}{2}})}. \end{aligned}$$

Using the standard iteration, we arrive

$$\|u\|_{L^q(B^{n-1}_{\frac{1}{4}})} \leq c(n,\alpha,p,q,r,a,b)(\|u\|_{L^p(B_1^{n-1})} + \|g\|_{L^q(B^{n-1}_{\frac{1}{2}})}).$$

□

Proof of Theorem 4.2. For $R > 0$, define

$$u_R(y) = \int_{\Sigma_R^n} \frac{v^\kappa(x)}{|x-y|^{n-\alpha}}dx, \quad v_R(x) = \int_{\Sigma_R^{n-1}} \frac{u^\theta(y)}{|x-y|^{n-\alpha}}dy.$$

Thus, from system (3.1), we have

$$\begin{aligned} u(y) &= \int_{B_R^+} \frac{v^\kappa(x)}{|x-y|^{n-\alpha}}dx + u_R(y), \\ v(x) &= \int_{B_R^{n-1}} \frac{u^\theta(y)}{|x-y|^{n-\alpha}}dy + v_R(x). \end{aligned}$$

We prove this theorem in two steps.

Step 1. We show: if $u \in L^{\theta+1}_{loc}(\partial\mathbb{R}^n_+)$, then $v \in L^{\kappa+1}_{loc}(\overline{\mathbb{R}^n_+})$, $v_R \in L^\infty_{loc}(B_R^+ \cup B_R^{n-1})$, and $u_R \in L^\infty_{loc}(B_R^{n-1})$.

We firstly show that if $u \in L^{\theta+1}_{loc}(\partial\mathbb{R}^n_+)$, then

$$v \in L^{\kappa+1}_{loc}(\overline{\mathbb{R}^n_+}) \quad \text{and} \quad v_R \in L^\infty_{loc}(B_R^+ \cup B_R^{n-1}).$$

In fact, since $u \in L^{\theta+1}_{loc}(\partial\mathbb{R}^n_+)$, we have $u < \infty$ a.e. in $\partial\mathbb{R}^n_+$. This implies $v < \infty$ a.e. in $\mathbb{R}^n_+$. Hence, there exists an $x_0 \in B^+_{\frac{R}{2}}$, such that $v(x_0) < \infty$, that is,

$$\int_{\Sigma_R^{n-1}} \frac{u^\theta(y)}{|y|^{n-\alpha}}dy \leq c\int_{\Sigma_R^{n-1}} \frac{u^\theta(y)}{|x_0-y|^{n-\alpha}}dy \leq cv(x_0) < \infty.$$

For $0 < \delta < 1, x \in B^+_{\delta R}$, it holds

$$v_R(x) \leq \frac{c(n,\alpha)}{(1-\delta)^{n-\alpha}}\int_{\Sigma_R^{n-1}} \frac{u^\theta(y)}{|y|^{n-\alpha}}dy.$$

This shows $v_R \in L^\infty_{loc}(B^+_R \cup B^{n-1}_R)$. On the other hand, using inequality (1.8) with $\frac{1}{\kappa+1} = \frac{n-1}{n}\big(\frac{n-\alpha}{n-1} - \frac{1}{\theta+1}\big)$, we have

$$[\int_{R^n_+} (\int_{B^{n-1}_R} \frac{u^\theta(y)}{|x-y|^{n-\alpha}} dy)^{\kappa+1} dx]^{\frac{1}{\kappa+1}} \le c\|u\|_{L^{\theta+1}(B^{n-1}_R)} < \infty.$$

This implies $v \in L^{\kappa+1}_{loc}(B^+_R \cup B^{n-1}_R)$. Since R is arbitrary, we have $v \in L^{\kappa+1}_{loc}(\overline{\mathbb{R}^n_+})$.

Next, we show $u_R \in L^\infty_{loc}(B^{n-1}_R)$. Indeed, since $u(y) < \infty$ a.e., there is a $y_0 \in B^{n-1}_{\frac{R}{2}}$, such that $u(y_0) < \infty$. Thus

$$\int_{\Sigma^n_R} \frac{v^\kappa(x)}{|x|^{n-\alpha}} dx \le c \int_{\Sigma^n_R} \frac{v^\kappa(x)}{|x-y_0|^{n-\alpha}} dx \le cu(y_0) < \infty.$$

For $0 < \delta < 1, y \in B^{n-1}_{\delta R}$, we have

$$u_R(y) \le \frac{c(n,\alpha)}{(1-\delta)^{n-\alpha}} \int_{\Sigma^n_R} \frac{v^\kappa(v)}{|x|^{n-\alpha}} dx < \infty.$$

That is, $u_R \in L^\infty_{loc}(B^{n-1}_R)$.

Step 2. We show $u \in C^\infty(\partial\mathbb{R}^n_+)$ and $v \in C^\infty(\overline{\mathbb{R}^n_+})$. To do this, we discuss two cases.

Case 1. $\frac{\alpha-1}{n-\alpha} < \theta < \frac{n+\alpha-2}{n-\alpha}$.

Since $\theta < \frac{n+\alpha-2}{n-\alpha}$, we can see from (4.1), that $\kappa > \frac{n+\alpha}{n-\alpha}$ and

$$(\theta+1) + \frac{(n-1)(\kappa+1)}{n} = \big(1 - \frac{\alpha}{n}\big)(\kappa+1)(\theta+1).$$

That is,

$$\kappa\theta - \frac{\alpha}{n}(\kappa+1)\theta - 1 = \frac{\alpha-1}{n}(\kappa+1) > 0.$$

Thus, we have

$$\big(\kappa - \frac{\alpha}{n}(\kappa+1)\big)\theta > 1.$$

This implies $\kappa - \frac{\alpha}{n}(\kappa+1) > \frac{1}{\theta}$. On the other hand, since $\kappa > \frac{n+\alpha}{n-\alpha}$, we have $\kappa - \frac{\alpha}{n}(\kappa+1) > 1$. Hence, we can choose a fixed number r such that

$$1 < \kappa - \frac{\alpha}{n}(\kappa+1) \le r \le \kappa, \quad \text{and} \quad r > \frac{1}{\theta}.$$

We have

$$u^{\frac{1}{r}}(y) \quad \le \quad (\int_{B^+_R} \frac{v^\kappa(x)}{|x-y|^{n-\alpha}} dx)^{\frac{1}{r}} + u^{\frac{1}{r}}_R(y),$$

which yields

$$\begin{aligned} v(x) &= \int_{B^{n-1}_R} \frac{u^{\theta-\frac{1}{r}}(y)u^{\frac{1}{r}}(y)}{|x-y|^{n-\alpha}} dy + v_R(x) \\ &\le \int_{B^{n-1}_R} \frac{u^{\theta-\frac{1}{r}}(y)}{|x-y|^{n-\alpha}} (\int_{B^+_R} \frac{v^{\kappa-r}(x)v^r(x)}{|x-y|^{n-\alpha}} dx)^{\frac{1}{r}} dy + h_R(x), \end{aligned}$$

where

$$h_R(x) = \int_{B^{n-1}_R} \frac{u^{\theta-\frac{1}{r}}(y)u^{\frac{1}{r}}_R(y)}{|x-y|^{n-\alpha}} dy + v_R(x).$$

Since $u_R \in L^{\infty}_{loc}(\partial\mathbb{R}^n_+)$, for any $x \in B^+_R$, we have

$$\int_{B_R^{n-1}} \frac{u^{\theta-\frac{1}{r}}(y)u_R^{\frac{1}{r}}(y)}{|x-y|^{n-\alpha}}dy \quad \leq \quad \|u_R\|^{\frac{1}{r}}_{L^\infty(B_R^{n-1})}\int_{B_R^{n-1}} \frac{u^{\theta-\frac{1}{r}}(y)}{|x-y|^{n-\alpha}}dy.$$

Note $u \in L^{\theta+1}_{loc}(\partial\mathbb{R}^n_+)$. We know, via inequality (1.8), that $h_R \in L^{q_0}(B^+_R \cup B^{n-1}_R)$ with q_0 given by

$$\begin{aligned}\frac{1}{q_0} &= \frac{n-1}{n}(\frac{\theta-1/r}{\theta+1} - \frac{\alpha-1}{n-1}) \\ &= \frac{n-1}{n}(\frac{n-\alpha}{n-1} - \frac{1+1/r}{\theta+1}) \\ &= \frac{n-1}{n}(\frac{n-\alpha}{n-1} - \frac{1}{\theta+1}) - \frac{n-1}{rn(\theta+1)} \\ &= \frac{1}{\kappa+1} - \frac{n-1}{rn(\theta+1)}.\end{aligned}$$

For $\varepsilon > 0$ small enough, we can choose $r = \kappa - \frac{\alpha}{n}(\kappa+1) + \varepsilon > 1 + \varepsilon$ so that

$$\begin{aligned}q_0 &= (\frac{1}{\kappa+1} - \frac{n-1}{rn(\theta+1)})^{-1} \\ &= \frac{rn(\kappa+1)}{n(r+1)-(n-\alpha)(\kappa+1)} \\ &= \frac{rn(\kappa+1)}{(n-\alpha)(\kappa+1)+n\varepsilon-(n-\alpha)(\kappa+1)} \\ &= \frac{rn(\kappa+1)}{n\varepsilon} > \frac{\kappa+1}{\varepsilon}\end{aligned}$$

can be any large number when we choose ε small enough. In the above derivation (in the second equality), we used the equation: $\frac{1}{\theta+1} = \frac{n-\alpha}{n-1} - \frac{n}{n-1}\frac{1}{\kappa+1}$, which can be deduced from (4.1) easily. Hence, it follows that $h_R \in L^q(B^+_R \cup B^{n-1}_R)$ for any $q < \infty$.

Now, in Proposition 4.3, take

$$\begin{aligned}U(y) &= u^{\theta-\frac{1}{r}}(y), \quad V(x) = v^{\kappa-r}(x), \\ a &= \frac{\kappa+1}{\kappa-r}, \quad b = \frac{\theta+1}{\theta-\frac{1}{r}}, \quad p = \kappa+1 > \frac{n}{n-\alpha}.\end{aligned}$$

Since $u \in L^{\theta+1}_{loc}(\partial\mathbb{R}^n_+)$ and $v \in L^{\kappa+1}_{loc}(\overline{\mathbb{R}^n_+})$, we have $U \in L^b(B^{n-1}_R)$ and $V \in L^a(B^+_R)$. Moreover, it is easy to verify via (4.1), that

$$\frac{n}{ra} + \frac{n-1}{b} = \frac{\alpha}{r} + (\alpha-1), \quad \text{and} \quad \frac{r}{p} + \frac{1}{a} = \frac{\kappa}{\kappa+1} < 1.$$

For

$$\kappa + 1 < q < \infty,$$

it is obvious that $\frac{r}{q}+\frac{1}{a}>\frac{\alpha}{n}$. We know from Proposition 4.3 that $v|_{B^+_{\frac{R}{4}}}\in L^q(B^+_{\frac{R}{4}})$ for small enough R. Hence, we can choose a q satisfying $n\kappa/\alpha<q<\infty$ such that

$$\begin{aligned}
u(y) &= \int_{B^+_{\frac{R}{4}}}\frac{v^\kappa(x)}{|x-y|^{n-\alpha}}dx+u_{\frac{R}{4}}(y),\\
&\le \Big(\int_{B^+_{\frac{R}{4}}}|x-y|^{\frac{(\alpha-n)q}{q-\kappa}}dx\Big)^{\frac{q-\kappa}{q}}\|v\|^\kappa_{L^q(B^+_{\frac{R}{4}})}+u_{\frac{R}{4}}(y)\\
&\le c(n,\alpha,q)R^{\frac{n(q-\kappa)}{q}-(n-\alpha)}\|v\|^\kappa_{L^q(B^+_{\frac{R}{4}})}+u_{\frac{R}{4}}(y)<\infty.
\end{aligned}$$

This also implies $u|_{\frac{R}{8}}\in L^\infty(B^{n-1}_{\frac{R}{8}})$. Since every point can be viewed as a center, we have $u\in L^\infty_{loc}(\partial\mathbb{R}^n_+)$, and hence $v\in L^\infty_{loc}(\overline{\mathbb{R}^n_+})$.

For any $R>0$, since

$$\int_{\Sigma^{n-1}_R}\frac{u^\theta(y)}{|y|^{n-\alpha}}dy<\infty,\quad \int_{\Sigma^n_R}\frac{v^\kappa(x)}{|x|^{n-\alpha}}dx<\infty,$$

we know $v_R\in C^\infty(B^+_R\cup B^{n-1}_R)$ and $u_R\in C^\infty(B^{n-1}_R)$. The first derivative of $\int_{B^+_R}\frac{v^\kappa(x)}{|x-y|^{n-\alpha}}dx$ is at least Hölder continuous in B^{n-1}_R (since $\alpha>1$). Since R is arbitrary, we know u is $C^{1,\tau}$ continuous on $\partial\mathbb{R}^n_+$, and hence v is $C^{1,\tau}$ continuous on $\partial\mathbb{R}^n_+$. Direct computation also shows v is $C^{1,\tau}$ continuous in $\overline{\mathbb{R}^n_+}$. By bootstrap, we conclude that $u\in C^\infty(\partial\mathbb{R}^n_+)$ and $v\in C^\infty(\overline{\mathbb{R}^n_+})$.

Case 2. For $\frac{n+\alpha-2}{n-\alpha}\le\theta<\infty$. In this case, from (4.1), it is easy to check $\frac{\alpha}{n-\alpha}<\kappa\le\frac{n+\alpha}{n-\alpha}$, and

$$(\kappa+1)+\frac{n(\theta+1)}{n-1}=\big(1-\frac{\alpha-1}{n-1}\big)(\kappa+1)(\theta+1).$$

That is,

$$\kappa\theta-\frac{\alpha-1}{n-1}(\theta+1)\kappa-1=\frac{\alpha}{n-1}(\theta+1)>0.$$

Thus, we have

$$\big(\theta-\frac{\alpha-1}{n-1}(\theta+1)\big)\kappa>1.$$

This implies $\theta-\frac{\alpha-1}{n-1}(\theta+1)>\frac{1}{\kappa}$. On the other hand, since $\theta\ge\frac{n+\alpha-2}{n-\alpha}$, we have $\theta-\frac{\alpha-1}{n-1}(\theta+1)\ge1$. Hence, we can choose a fixed number r such atht

$$1\le\theta-\frac{\alpha-1}{n-1}(\theta+1)\le r\le\theta,\quad\text{and}\quad r>\frac{1}{\kappa},$$

and then

$$v^{\frac{1}{r}}(x)\quad\le\quad\Big(\int_{B^{n-1}_R}\frac{u^\theta(y)}{|x-y|^{n-\alpha}}dy\Big)^{\frac{1}{r}}+v_R^{\frac{1}{r}}(x).$$

Thus,

$$u(y)\quad=\quad\int_{B^+_R}\frac{v^{\kappa-\frac{1}{r}}(x)}{|x-y|^{n-\alpha}}\Big(\int_{B^{n-1}_R}\frac{u^\theta(y)}{|x-y|^{n-\alpha}}dy\Big)^{\frac{1}{r}}dx+g_R(y),$$

where

$$g_R(y) = \int_{B_R^+} \frac{v^{\kappa-\frac{1}{r}}(x) v_R^{\frac{1}{r}}(x)}{|x-y|^{n-\alpha}} dx + u_R(y).$$

For any $y \in B_R^{n-1}$,

$$\int_{B_R^+} \frac{v^{\kappa-\frac{1}{r}}(x) v_R^{\frac{1}{r}}(x)}{|x-y|^{n-\alpha}} dx \le \|v_R\|_{L^\infty(B_R^+)}^{\frac{1}{r}} \int_{B_R^+} \frac{v^{\kappa-\frac{1}{r}}(x)}{|x-y|^{n-\alpha}} dx$$

By inequality (1.12), we have $g_R \in L^{q_1}(B_R^{n-1})$ with $\theta+1 < q_1 \le \infty$, where $q_1 = \frac{(n-1)r(\theta+1)}{(n-1)(r+1)-(n-\alpha)(\theta+1)}$. As in case 1: q_1 can be chosen as any larger number.

Now in Proposition 4.4, take

$$U(y) = u^{\theta-r}(y), \quad V(x) = v^{\kappa-\frac{1}{r}}(x),$$
$$a = \frac{\theta+1}{\theta-r} \quad b = \frac{\kappa+1}{\kappa-\frac{1}{r}}, \quad p = \theta+1.$$

Since $u \in L_{loc}^{\theta+1}(\partial\mathbb{R}_+^n)$ and $v \in L_{loc}^{\kappa+1}(\overline{\mathbb{R}_+^n})$, we have $U \in L^a(B_R^{n-1})$ and $V \in L^b(B_R^+)$, and it is easy to verify that

$$\frac{n-1}{ar} + \frac{n}{b} = \frac{\alpha-1}{r} + \alpha, \quad \frac{r}{p} + \frac{1}{a} = \frac{\theta}{\theta+1} < 1,$$

For any $\theta+1 < q < \infty$, it follows from Proposition 4.4 that $u \in L^q(B_R^{n-1})$. Since every point can be viewed as a center, we have $u \in L_{loc}^q(\partial\mathbb{R}_+^n)$.

Now, similar to the argument in previous case, we have $v \in L_{loc}^\infty(\overline{\mathbb{R}_+^n})$, and then $u \in L_{loc}^\infty(\partial\mathbb{R}_+^n)$. It follows that $u \in C^\infty(\partial\mathbb{R}_+^n)$ and $v \in C^\infty(\overline{\mathbb{R}_+^n})$.

□

5. Miscellaneous

In this section, we shall include some related results concerning the computation of the sharp constants, operators on a bounded domain, inequality for limit case ($\alpha = n$), fractional Laplacian operators and some non-existence to a system of integral equations.

5.1. Integral inequalities in a bounded domain.

For a smooth and bounded domain $\Omega \subset \mathbb{R}^n$, we introduce the following operators

$$\widetilde{E_\alpha} f(x) = \int_{\partial\Omega} \frac{f(y)}{|x-y|^{n-\alpha}} dS_y, \ \forall\, x \in \Omega \tag{5.1}$$

and

$$\widetilde{R_\alpha} g(y) = \int_\Omega \frac{g(x)}{|x-y|^{n-\alpha}} dV_x, \ \forall\, y \in \partial\Omega. \tag{5.2}$$

From Theorem 1.1, we first show

Corollary 5.1. *Assume $1 < \alpha < n$. For any $f \in L^{\frac{2n}{n+\alpha-2}}(\partial B_R), g \in L^{\frac{2n}{n+\alpha}}(B_R)$ with $R > 0$,*

$$\|\widetilde{E_\alpha} f\|_{L^{\frac{2n}{n-\alpha}}(B_R)} \le C_e(n, \alpha, \frac{2(n-1)}{n+\alpha-2}) \|f\|_{L^{\frac{2(n-1)}{n+\alpha-2}}(\partial B_R)}, \tag{5.3}$$

$$\|\widetilde{R_\alpha} g\|_{L^{\frac{2(n-1)}{n-\alpha}}(\partial B_R)} \le C_r(n, \alpha, \frac{2n}{n+\alpha}) \|g\|_{L^{\frac{2n}{n+\alpha}}(B_R)}. \tag{5.4}$$

Moreover the best constants $C_e(n,\alpha,\frac{2(n-1)}{n+\alpha-2})$ *is given by*

$$C_e(n,\alpha,\frac{2(n-1)}{n+\alpha-2}) = (n\omega_n)^{-\frac{n+\alpha-2}{2(n-1)}}\Big(\int_{B_1(x_1)}\Big(\int_{\partial B_1(x_1)}\frac{1}{|\xi-z|^{n-\alpha}}dS_z\Big)^{\frac{2n}{n-\alpha}}d\xi\Big)^{\frac{n-\alpha}{2n}}.$$

where $x_1=(0,-1)$, *and* $C_r(n,\alpha,\frac{2n}{n+\alpha})=C_e(n,\alpha,\frac{2(n-1)}{n+\alpha-2})$. *In particular, for* $\alpha=2$,

$$C_e(n,2,\frac{2(n-1)}{n})=C_r(n,2,\frac{2n}{n+2})=n^{\frac{n-2}{2(n-1)}}\omega_n^{1-\frac{1}{n}-\frac{1}{2(n-1)}}. \tag{5.5}$$

Proof. For given $\mu\neq 0$, define

$$\omega_{\mu,x_0,\lambda}(y)=(\frac{\lambda}{|y-x_0|})^{\mu}\omega(y^{x_0,\lambda}),\quad y^{x_0,\lambda}=\frac{\lambda^2(y-x_0)}{|y-x_0|^2}+x_0,$$

where $y\in\overline{\mathbb{R}^n_+}$, $x_0=(x_0',-\lambda)$, $\lambda>0$. Directly computation shows

$$\begin{aligned}\int_{\partial\mathbb{R}^n_+}|f(y)|^{\frac{2(n-1)}{n+\alpha-2}}dy &= \int_{\partial B_{\frac{\lambda}{2}}(x_1)}|f(z^{x_0,\lambda})|^{\frac{2(n-1)}{n+\alpha-2}}(\frac{\lambda}{|y-x_0|})^{2(n-1)}dS_z \quad (\text{let } y=z^{x_0,\lambda})\\ &= \int_{\partial B_{\frac{\lambda}{2}}(x_1)}|f_{\mu,x_0,\lambda}(z)|^{\frac{2(n-1)}{n+\alpha-2}}(\frac{\lambda}{|y-x_0|})^{2(n-1)-\mu\frac{2(n-1)}{n+\alpha-2}}dS_z,\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^n_+}|E_\alpha f(x)|^{\frac{2n}{n-\alpha}}dx\\ =&\int_{B_{\frac{\lambda}{2}}(x_1)}\Big|\int_{\partial B_{\frac{\lambda}{2}}(x_1)}\frac{f(z^{x_0,\lambda})}{|\xi^{x_0,\lambda}-z^{x_0,\lambda}|^{n-\alpha}}(\frac{\lambda}{|y-x_0|})^{2(n-1)}dS_z\Big|^{\frac{2n}{n-\alpha}}(\frac{\lambda}{|\xi-x_0|})^{2n}d\xi\\ =&\int_{B_{\frac{\lambda}{2}}(x_1)}\Big|\int_{\partial B_{\frac{\lambda}{2}}(x_1)}\frac{f_{\mu,x_0,\lambda}(z)}{|\xi-z|^{n-\alpha}}(\frac{\lambda}{|y-x_0|})^{\gamma_1}dS_z\Big|^{\frac{2n}{n-\alpha}}(\frac{\lambda}{|\xi-x_0|})^{\gamma_2}d\xi,\end{aligned}$$

where $x_1=(x_0,-\lambda/2)$, $\gamma_1=2(n-1)-\mu-(n-\alpha)$, and $\gamma_2=2n-(n-\alpha)\frac{2n}{n-\alpha}=0$.

In the case of $\mu=n+\alpha-2$, we will denote $f_{\mu,x,\lambda}$ as $f_{x,\lambda}$. Thus our notation is consistent with those in Section 3. In fact, it is clear from the above that for $\mu=n+\alpha-2$,

$$\|f\|_{L^{\frac{2(n-1)}{n+\alpha-2}}(\partial\mathbb{R}^n_+)}=\|f_{x_0,\lambda}\|_{L^{\frac{2(n-1)}{n+\alpha-2}}(\partial B_{\frac{\lambda}{2}}(x_1))}$$

and

$$\|E_\alpha f\|_{L^{\frac{2n}{n-\alpha}}(\mathbb{R}^n_+)}=\|\widetilde{E_\alpha}(f_{x_0,\lambda})\|_{L^{\frac{2n}{n-\alpha}}(B_{\frac{\lambda}{2}}(x_1))}.$$

Note: $(f_{x_0,\lambda})_{x_0,\lambda}=f$, inequality (5.3) follows from inequality (1.8).

In the case of $\mu=n+\alpha$, we have

$$\|g\|_{L^{\frac{2n}{n+\alpha}}(\mathbb{R}^n_+)}=\|g_{\mu,x_0,\lambda}\|_{L^{\frac{2n}{n+\alpha}}(B_{\frac{\lambda}{2}}(x_1))}$$

and

$$\|R_\alpha g\|_{L^{\frac{2(n-1)}{n-\alpha}}(\partial\mathbb{R}^n_+)}=\|\widetilde{R_\alpha}(g_{\mu,x_0,\lambda})\|_{L^{\frac{2(n-1)}{n-\alpha}}(\partial B_{\frac{\lambda}{2}}(x_1))}.$$

Inequality (5.4) follows from inequality (1.12).

Now we compute the best constant $C_e(n,\alpha,p)$. From Theorem 1.4, we know that

$$f(y)=(\frac{\lambda}{|y-x_0|})^{n+\alpha-2},\quad \forall\, y\in\partial\mathbb{R}^n_+$$

is an extremal function to inequality (1.8) for any $\lambda>0$ and $x_0=(0,-\lambda)$. Let $x_1=(0,-\lambda/2)$ and $\mu=n+\alpha-2$. Then for $y\in\partial B_{\frac{\lambda}{2}}(x_1)$,

$$\begin{aligned} f_{\mu,x_0,\lambda}(y) &= (\frac{\lambda}{|y-x_0|})^{n+\alpha-2}(\frac{\lambda}{|y^{x_0,\lambda}-x_0|})^{n+\alpha-2} \\ &= (\frac{\lambda}{|y-x_0|})^{n+\alpha-2}(\frac{|y-x_0|}{\lambda})^{n+\alpha-2}=1; \end{aligned}$$

And,

$$\begin{aligned} \|\widetilde{E_\alpha}(f_{\mu,x_0,\lambda})\|^{\frac{2n}{n-\alpha}}_{L^{\frac{2n}{n-\alpha}}(B_{\frac{\lambda}{2}}(x_1))} &= \int_{B_{\frac{\lambda}{2}}(x_1)}\Big(\int_{\partial B_{\frac{\lambda}{2}}(x_1)}\frac{f_{\mu,x_0,\lambda}(z)}{|\xi-z|^{n-\alpha}}dS_z\Big)^{\frac{2n}{n-\alpha}}d\xi \\ &= \int_{B_{\frac{\lambda}{2}}(x_1)}\Big(\int_{\partial B_{\frac{\lambda}{2}}(x_1)}\frac{1}{|\xi-z|^{n-\alpha}}dS_z\Big)^{\frac{2n}{n-\alpha}}d\xi \\ &= (\frac{\lambda}{2})^{\frac{n+\alpha-2}{2}\cdot\frac{2n}{n-\alpha}}\int_{B_1(x_2)}\Big(\int_{\partial B_1(x_2)}\frac{1}{|\xi-z|^{n-\alpha}}dS_z\Big)^{\frac{2n}{n-\alpha}}d\xi, \end{aligned}$$

where $x_2=\big(\frac{\lambda}{2}\big)^{-1}x_1=(0,-1)$. On the other hand,

$$\begin{aligned} \|f\|_{L^{\frac{2(n-1)}{n+\alpha-2}}(\partial\mathbb{R}^n_+)} &= \|f_{\mu,x_0,\lambda}\|_{L^{\frac{2(n-1)}{n+\alpha-2}}(\partial B_{\frac{\lambda}{2}}(x_1))} \\ &= \Big(\int_{\partial B_{\frac{\lambda}{2}}(x_1)}dS_z\Big)^{\frac{n+\alpha-2}{2(n-1)}} \\ &= \Big(n\omega_n(\frac{\lambda}{2})^{n-1}\Big)^{\frac{n+\alpha-2}{2(n-1)}}. \end{aligned}$$

Thus, we have

$$\begin{aligned} C_e(n,\alpha,\frac{2(n-1)}{n+\alpha-2}) &= \Big(\int_{B_{\frac{\lambda}{2}}(x_1)}|E_\alpha f_{\mu,x_0,\lambda}(\xi)|^{\frac{2n}{n-\alpha}}d\xi\Big)^{\frac{n-\alpha}{2n}}\cdot\|f_{\mu,x_0,\lambda}\|^{-1}_{L^{\frac{2(n-1)}{n+\alpha-2}}(\partial B_{\frac{\lambda}{2}}(x_1))} \\ &= (n\omega_n)^{-\frac{n+\alpha-2}{2(n-1)}}\Big(\int_{B_1(x_2)}\Big(\int_{\partial B_1(x_2)}\frac{1}{|\xi-z|^{n-\alpha}}dS_z\Big)^{\frac{2n}{n-\alpha}}d\xi\Big)^{\frac{n-\alpha}{2n}}. \end{aligned}$$

For general $\alpha>1$, it is not easy to obtain the precise value for the sharp constant. However, for $\alpha=2$, we can identify it.

Observe: for $\alpha=2$, if $\lambda=2$, $x_0=(0,-\lambda)$, $x_1=(x_0,-\lambda/2)$, then $f_{x_0,\lambda}=1$ on $\partial B_{\frac{\lambda}{2}}(x_1)$ and $\widetilde{E_2}(f_{x_0,\lambda})(\xi)$ is a harmonic function in $B_{\frac{\lambda}{2}}(x_1)$. Thus $\widetilde{E_2}(f_{x_0,\lambda})(\xi)$ is a constant, in particular

$$\widetilde{E_2}(f_{x_0,\lambda})(\xi)=\widetilde{E_2}(f_{x_0,\lambda})(\xi)|_{\xi=x_1}=n\omega_n.$$

It follows that

$$C_e(n,2,\frac{2(n-2)}{n})=\omega_n^{1-\frac{1}{n}-\frac{1}{2(n-1)}}n^{\frac{n-2}{2(n-1)}}.$$

□

We remark here that for $\alpha \neq 2$, $\widetilde{E_\alpha}(1)$ may not be constant in the ball $B_{\frac{\lambda}{2}}(x_1)$, thus solution $E_\alpha(f)(x)$ to (1.13) may not be, up to a constant multiplier, in the form of

$$(\frac{1}{|x'-x_0|^2+(x_n+\lambda)^2})^{\frac{n+\alpha-2}{2}}. \tag{5.6}$$

For example, for $\alpha=4$ and $n\geq 5$, let $U_4(x)=\widetilde{E_\alpha}(1)$. Use the same notations as those in the proof of Corollary 5.1, then for $x\in B_{\frac{\lambda}{2}}(x_1)$,

$$V(x):=\Delta_x U_4(x)=2(4-n)\int_{\partial B_{\frac{\lambda}{2}}(x_1)}\frac{1}{|x-z|^{n-2}}dz.$$

Clearly V satisfies

$$\begin{cases}\Delta_x V(x)=0, & \text{in } B_{\frac{\lambda}{2}}(x_1),\\ V=c_1, & \text{on } \partial B_{\frac{\lambda}{2}}(x_1).\end{cases} \tag{5.7}$$

If follows that $V\equiv c_1$ in $B_{\frac{\lambda}{2}}(x_1)$. It implies

$$U_4(x)=c_1+c_2|x-x_1|^2.$$

for some constant c_1 and $c_2\neq 0$. Thus for an extremal function $f(y)$ (so that $U_4(x)=f_{x_1,\lambda}$),

$$E_4(f)(x)=c\int_{\partial\mathbb{R}^n_+}(\frac{1}{|y-y_0|^2+\lambda^2})^{\frac{n+2}{2}}\cdot\frac{1}{|x-y|^{n-4}}dy,$$

is not in the form of (5.6).

As a simple consequence, we show inequality (1.12) implies standard trace inequality with p-biharmonic operator.

Corollary 5.2. *For $n>2$ and $p\in(1,n/2)$, and $q=p(n-1)/(n-2p)$, there is a constant $C(p,n)>0$, such that, for all $f\in W^{2,p}(\mathbb{R}^n_+)$,*

$$\|f\|_{L^q(\partial\mathbb{R}^n_+)}\leq C(p,n)\|\Delta f\|_{L^p(\mathbb{R}^n_+)}.$$

To this end, we need the following representation formula.

Lemma 5.3. *For any $f\in C_0^\infty(\mathbb{R}^n_+)$ and $x\in\partial\mathbb{R}^n_+$,*

$$f(x)=\frac{1}{C(n)}\int_{\mathbb{R}^n_+}\frac{\langle\nabla f(y),x-y\rangle}{|x-y|^n}dy=\frac{1}{(2-n)C(n)}\int_{\mathbb{R}^n_+}\frac{\Delta f(y)}{|x-y|^{n-2}}dy,$$

where $C(n)=n\omega_n/2$ is the half of the surface area of the unit sphere S^n.

Proof. Let $z\in\partial B_1^+(x)$ and write

$$f(x)=-\int_0^\infty\frac{d}{dt}f(x+tz)dt=-\int_0^\infty\langle\nabla f(x+tz),z\rangle dt.$$

Integrating both sides on $\partial B_1^+(x)$ with respect to z variable, we have

$$\begin{aligned}C(n)f(x) &= -\int_{\partial B_1^+(x)}\int_0^\infty\langle\nabla f(x+tz),z\rangle dtdS_z\\ &= \int_{\mathbb{R}^n_+}\frac{\langle\nabla_y f(y),x-y\rangle}{|x-y|^n}dy.\end{aligned}$$

Further, using integrating by part, we have

$$\begin{aligned}\int_{\mathbb{R}^n_+} \frac{\langle \nabla_y f(y), x-y\rangle}{|x-y|^n} dy &= \frac{1}{n-2}\int_{\mathbb{R}^n_+} \langle \nabla_y f(y), \nabla_y(|x-y|^{2-n})\rangle dy \\ &= -\frac{1}{n-2}\int_{\mathbb{R}^n_+} \frac{\Delta f(y)}{|x-y|^{n-2}} dy.\end{aligned}$$

□

Proof of corollary 5.2. From Lemma 5.3, we have for any $f \in C_0^\infty(\mathbb{R}^n_+)$,

$$|f(x)| \le \frac{1}{(n-2)C(n)} \int_{\mathbb{R}^n_+} \frac{|\Delta f(y)|}{|x-y|^{n-2}} dy.$$

Let $\frac{1}{q} = \frac{n}{n-1}\left(\frac{1}{p} - \frac{2}{n}\right)$. It follows from the above and (1.12) that

$$\|f\|_{L^q(\partial\mathbb{R}^n_+)} \le C(p,n)\|\Delta f\|_{L^p(\mathbb{R}^n_+)}.$$

5.2. **Inequality in limit case $\alpha = n$.** Inequality (5.3) is equivalent to the following inequality

$$\int_{B_1}\int_{\partial B_1} \frac{G(\xi)F(\eta)}{|\xi-\eta|^{n-\alpha}} dS_\eta d\xi \le C_e(n,\alpha,p)\|F\|_{L^p(\partial B_1)}\|G\|_{L^t(B_1)}, \tag{5.8}$$

with $p = \frac{2(n-1)}{n+\alpha-2}$ and $t = \frac{2n}{n+\alpha}$.

Standard limiting argument for $\alpha \to n^-$ (see, for example, Beckner [2]) and tedious computations yield

Corollary 5.4. *Assume that F and G are nonnegative $L\ln L$ functions on ∂B_1 and B_1, respectively, with $\int_{\partial B_1} F(\eta)dS_\eta = \int_{B_1} G(\xi)d\xi = 1$. Then*

$$\begin{aligned}&-2n\omega_n \int_{B_1}\int_{\partial B_1} G(\xi)\ln|\xi-\eta|F(\eta)dS_\eta d\xi \\ \le\ & \frac{1}{n}\int_{B_1} G(\xi)\ln(G(\xi))d\xi + \frac{1}{n-1}\int_{\partial B_1} F(\eta)\ln(F(\eta))dS_\eta + C_n,\end{aligned}$$

where $C_n = \frac{\ln(n\omega_n)}{n-1} + \frac{1}{n}\ln\int_{B_1} e^{I_n(\xi)}d\xi$ and $I_n(\xi) = -2\omega_n^{-1}\int_{\partial B_1}\ln|\xi-\eta|dS_\eta$.

Equivalently, assume that $f(y)$ and $g(x)$ are nonnegative $L\ln L$ functions on $\partial\mathbb{R}^n_+$ and $\mathbb{R}^n_+$, respectively, with $\int_{\partial\mathbb{R}^n_+} f(y)dy = \int_{\mathbb{R}^n_+} g(x)dx = 1$. Then

$$\begin{aligned}&-2n\omega_n \int_{\mathbb{R}^n_+}\int_{\partial\mathbb{R}^n_+} g(x)\ln|x-y|f(y)dydx \\ \le\ & \frac{1}{n}\int_{\mathbb{R}^n_+} g(x)\ln(g(x))dx + \frac{1}{n-1}\int_{\partial\mathbb{R}^n_+} f(y)\ln(f(y))dy + C_n.\end{aligned} \tag{5.9}$$

5.3. **Fractional Lapalacian operators.** Let

$$u(x) = \frac{1}{c(n,\alpha)}\int_{\partial\mathbb{R}^n_+} \frac{f(y)}{|x-y|^{n-\alpha}} dy. \tag{5.10}$$

Direct computation shows that $u(x)$ satisfies the following equation for $\alpha = 2m$.

Proposition 5.5. *Assume that* $f(y) \in C^{\infty}(\mathbb{R}^{n-1})$. *If* $u(x)$ *satisfies* (5.10) *for* $\alpha = 2m$ $(m \in \mathbb{Z})$, *then*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = 0 & \text{for } x_n > 0 \text{ and } x' \in \mathbb{R}^{n-1}, \\ \frac{\partial}{\partial x_n}[(-\Delta)^k u] = 0, \frac{\partial}{\partial x_n}[(-\Delta)^{m-1} u] = (-1)^m f(x'), & x' \in \mathbb{R}^{n-1}, \end{cases} \tag{5.11}$$

where $k = 0, 1, 2, 3, \cdots, m-2$.

Note that

$$c(n-1, \alpha-1)^{-1} E_\alpha f(x', 0) = I_\alpha f(x'). \tag{5.12}$$

As a simple consequence, we know that the global defined equation

$$(-\Delta)^{\frac{\beta}{2}} u(x) = f(x), \quad x \in \mathbb{R}^{n-1}$$

is equivalent to a pointwise defined equation (5.11), where $\beta = \alpha - 1$. In particular, for $\beta = 1$ (i.e. $\alpha = 2$), the above equivalent relation is a well known fact, and was pointed out in, e.g. Caffarelli and Silverster [8], and Cabre and Tan [7], and others.

5.4. **Nonexistence of positive solutions to the integral system with subcritical powers.** As a byproduct of using method of moving sphere, we will show system (3.1) with subcritical powers:

$$1 < \theta \le \frac{n+\alpha-2}{n-\alpha}, \;\; 1 < \kappa \le \frac{n+\alpha}{n-\alpha}, \text{ and } \theta + \kappa < \frac{2n+2\alpha-2}{n-\alpha} \tag{5.13}$$

has only a pair of trivial non-negative solutions. Note that $\theta + \kappa = \frac{2n+2\alpha-2}{n-\alpha}$ is equivalent to the critical case

$$\frac{1}{\theta+1} + \frac{n}{(\kappa+1)(n-1)} = \frac{n-\alpha}{n-1},$$

which is a necessary condition for inequality (1.8).

Theorem 5.6. *Let* $1 < \alpha < n$. *Suppose* $(u, v) \in L_{loc}^{\frac{(n-1)(\theta-1)}{\alpha-1}}(\partial\mathbb{R}^n_+) \times L_{loc}^{\frac{n(\kappa-1)}{\alpha}}(\overline{\mathbb{R}^n_+})$ *is a pair of non-negative solutions to system* (3.1) *with* θ, κ *satisfying (5.13). Then* $u = v = 0$.

Note (5.13) is equivalent to

$$\frac{2(n-1)}{n+\alpha-2} \le p < 2, \, 2 < q \le \frac{2n}{n-\alpha}, \text{ and } \frac{1}{p} - \frac{n}{q(n-1)} < \frac{\alpha-1}{n}.$$

Theorem 5.6 yields the proof for Theorem 1.5.

To prove the theorem, we first show

Lemma 5.7. *Under the same assumptions in Theorem 5.6, if* (u, v) *is a pair of nonnegative solutions to system* (3.1), *then for any* $x \in \partial\mathbb{R}^n_+$, *there exists* $\lambda_0(x) > 0$ *such that :* $0 < \lambda < \lambda_0(x)$,

$$\begin{aligned} u_{x,\lambda}(\xi) &\le u(\xi), \quad \forall\, \xi \in \Sigma^{n-1}_{x,\lambda}, \\ v_{x,\lambda}(\eta) &\le v(\eta), \quad \forall\, \eta \in \Sigma^n_{x,\lambda}. \end{aligned}$$

Proof. The proof is similar to that of Lemma 3.3. Since $\kappa \le \frac{n+\alpha}{n-\alpha}$, it is obvious that $\tau_1 \ge 0$, and $\big(\frac{\lambda}{|\eta-x|}\big)^{\tau_1} \le 1$ for $\eta \in \Sigma^n_{x,\lambda}$. Thus, for any $\xi \in \Sigma^u_{x,\lambda}$, using (3.4) and mean value theorem it has

$$\begin{aligned} 0 \le u_{x,\lambda}(\xi) - u(\xi) &= \int_{\Sigma^n_{x,\lambda}} P(x,\lambda;\xi,\eta)\big[\big(\frac{\lambda}{|\xi-x|}\big)^{\tau_1} v^\kappa_{x,\lambda}(\eta) - v^\kappa(\eta)\big]d\eta, \\ &\le \int_{\Sigma^n_{x,\lambda}} \frac{v^\kappa_{x,\lambda}(\eta) - v^\kappa(\eta)}{|\xi-\eta|^{n-\alpha}} d\eta, \\ &\le \kappa \int_{\Sigma^v_{x,\lambda}} \frac{v^{\kappa-1}_{x,\lambda}(\eta)(v_{x,\lambda}(\eta) - v(\eta))}{|\xi-\eta|^{n-\alpha}} d\eta. \end{aligned}$$

For any $t \in (1, \frac{n}{\alpha})$, take $k = \frac{(n-1)t}{n-\alpha t}$, $s = \frac{nt}{n-\alpha t}$. Similar to Lemma 3.3, integrating on $\partial\mathbb{R}^n_+$ and using inequality (1.12) and Hölder inequality, from the above inequality, we have

$$\begin{aligned} \|u_{x,\lambda} - u\|_{L^k(\Sigma^u_{x,\lambda})} &\le c\|v^{\kappa-1}_{x,\lambda}(v_{x,\lambda} - v)\|_{L^t(\Sigma^v_{x,\lambda})} \\ &\le c\|v_{x,\lambda}\|^{\kappa-1}_{L^{\frac{n(\kappa-1)}{\alpha}}(\Sigma^v_{x,\lambda})} \|v_{x,\lambda} - v\|_{L^s(\Sigma^v_{x,\lambda})}. \end{aligned} \tag{5.14}$$

On the other hand, since $\theta \le \frac{n+\alpha-2}{n-\alpha}$, it holds $\tau_2 \ge 0$, and $\big(\frac{\lambda}{|\xi-x|}\big)^{\tau_2} \le 1$ for $\xi \in \Sigma^{n-1}_{x,\lambda}$. Thus, for any $\eta \in \Sigma^v_{x,\lambda}$, it follows from (3.5) that

$$v_{x,\lambda}(\eta) - v(\eta) \le \theta \int_{\Sigma^u_{x,\lambda}} \frac{u^{\theta-1}_{x,\lambda}(\xi)(u_{x,\lambda}(\xi) - u(\xi))}{|\eta-\xi|^{n-\alpha}} d\xi,$$

Since $s > \frac{n}{n-\alpha}$ for any $t \in (1, \frac{n}{\alpha})$, we can use HLS inequality (1.8) and Hölder inequality to arrive at

$$\begin{aligned} \|v_{x,\lambda} - v\|_{L^s(\Sigma^v_{x,\lambda})} &\le c\|u^{\theta-1}_{x,\lambda}(u_{x,\lambda} - u)\|_{L^{\frac{(n-1)s}{n+(\alpha-1)s}}(\Sigma^u_{x,\lambda})} \\ &\le c\|u_{x,\lambda}\|^{\theta-1}_{L^{\frac{(n-1)(\theta-1)}{\alpha-1}}(\Sigma^u_{x,\lambda})} \|u_{x,\lambda} - u\|_{L^k(\Sigma^u_{x,\lambda})}. \end{aligned} \tag{5.15}$$

Combining (5.14) with (5.15), we obtain

$$\begin{aligned} \|u_{x,\lambda} - u\|_{L^k(\Sigma^u_{x,\lambda})} &\le c\|v_{x,\lambda}\|^{\kappa-1}_{L^{\frac{n(\kappa-1)}{\alpha}}(\Sigma^v_{x,\lambda})} \|u_{x,\lambda}\|^{\theta-1}_{L^{\frac{(n-1)(\theta-1)}{\alpha-1}}(\Sigma^u_{x,\lambda})} \|u_{x,\lambda} - u\|_{L^k(\Sigma^u_{x,\lambda})} \\ &\le c\|v\|^{\kappa-1}_{L^{\frac{n(\kappa-1)}{\alpha}}(B^+_\lambda(x))} \|u\|^{\theta-1}_{L^{\frac{(n-1)(\theta-1)}{\alpha-1}}(B^{n-1}_\lambda(x))} \|u_{x,\lambda} - u\|_{L^k(\Sigma^u_{x,\lambda})}. \end{aligned} \tag{5.16}$$

Similarly, we have

$$\|v_{x,\lambda} - v\|_{L^s(\Sigma^v_{x,\lambda})} \le c\|v\|^{\kappa-1}_{L^{\frac{n(\kappa-1)}{\alpha}}(B^+_\lambda(x))} \|u\|^{\theta-1}_{L^{\frac{(n-1)(\theta-1)}{\alpha-1}}(B^{n-1}_\lambda(x))} \|v_{x,\lambda} - v\|_{L^s(\Sigma^v_{x,\lambda})}. \tag{5.17}$$

Since $u \in L^{\frac{(n-1)(\theta-1)}{\alpha-1}}_{loc}(\partial\mathbb{R}^n_+)$ and $v \in L^{\frac{n(\kappa-1)}{\alpha}}_{loc}(\mathbb{R}^n_+)$, we can choose λ_0 small enough such that $0 < \lambda < \lambda_0$, and

$$c\|v\|^{\kappa-1}_{L^{\frac{n(\kappa-1)}{\alpha}}(B^+_\lambda(x))} \|u\|^{\theta-1}_{L^{\frac{(n-1)(\theta-1)}{\alpha-1}}(B^{n-1}_\lambda(x))} \le \frac{1}{2}.$$

Combining the above with (5.16) and (5.17), we get

$$\begin{aligned} \|u_{x,\lambda} - u\|_{L^k(\Sigma^u_{x,\lambda})} &\le \frac{1}{2}\|u_{x,\lambda} - u\|_{L^k(\Sigma^u_{x,\lambda})}, \\ \|v_{x,\lambda} - v\|_{L^s(\Sigma^v_{x,\lambda})} &\le \frac{1}{2}\|v_{x,\lambda} - v\|_{L^s(\Sigma^v_{x,\lambda})}, \end{aligned}$$

which imply $\|u_{x,\lambda} - u\|_{L^k(\Sigma^u_{x,\lambda})} = \|v_{x,\lambda} - v\|_{L^s(\Sigma^v_{x,\lambda})} = 0$. Thus, both $\Sigma^u_{x,\lambda}$ and $\Sigma^v_{x,\lambda}$ have measure zero. We complete the proof of the lemma. □

Define

$$\bar{\lambda}(x) = sup\{\mu > 0 \,|\, u_{x,\lambda}(\xi) \le u(\xi), \text{and } v_{x,\lambda}(\eta) \le v(\eta),\ \forall \lambda \in (0,\mu), \forall \xi \in \Sigma^{n-1}_{x,\lambda}, \forall \eta \in \Sigma^n_{x,\lambda}\}.$$

We will show the sphere will never stop.

Lemma 5.8. *$\bar{\lambda}(x) = \infty$ for all $x \in \partial\mathbb{R}^n_+$.*

Proof. We prove it by contradiction argument. Suppose the contrary, there exists some $x_0 \in \partial\mathbb{R}^n_+$ such that $\bar{\lambda}(x_0) < \infty$. By the definition of $\bar{\lambda}$,

$$\begin{aligned} u_{x_0,\bar{\lambda}}(\xi) &\le u(\xi) \quad \text{for } \xi \in \Sigma^{n-1}_{x_0,\bar{\lambda}}, \\ v_{x_0,\bar{\lambda}}(\eta) &\le v(\eta) \quad \text{for } \eta \in \Sigma^n_{x_0,\bar{\lambda}}. \end{aligned}$$

From (3.4) and (3.5) with $x = x_0, \lambda = \bar{\lambda}$ and the fact that at least one of τ_1 and τ_2 is positive, we have

$$\begin{aligned} u_{x_0,\bar{\lambda}}(\xi) &< u(\xi) \quad \text{for } \xi \in \Sigma^{n-1}_{x_0,\bar{\lambda}}, \\ v_{x_0,\bar{\lambda}}(\eta) &< v(\eta) \quad \text{for } \eta \in \Sigma^n_{x_0,\bar{\lambda}}. \end{aligned}$$

Similar to proof process of Lemma 3.3, for $\lambda \in [\bar{\lambda}, \bar{\lambda} + \varepsilon)$, we can conclude that $\Sigma^u_{x_0,\lambda}$ and $\Sigma^v_{x_0,\lambda}$ must have measure zero. Thus we obtain that

$$\begin{aligned} u_{x_0,\lambda}(\xi) &\le u(\xi) \quad \text{for } \xi \in \Sigma^{n-1}_{x_0,\lambda}, \\ v_{x_0,\lambda}(\eta) &\le v(\eta) \quad \text{for } \eta \in \Sigma^n_{x_0,\lambda} \end{aligned}$$

for $\lambda \in [\bar{\lambda}, \bar{\lambda} + \varepsilon)$, which contradicts the definition of $\bar{\lambda}$. □

Proof of Theorem 5.6. According to Lemma 5.8, $\bar{\lambda}(x) = \infty$ for all $x \in \partial\mathbb{R}^n_+$, that is, for all $\lambda > 0$ and $x \in \partial\mathbb{R}^n_+$,

$$\begin{aligned} u_{x,\lambda}(\xi) &\le u(\xi) \quad \text{for } \xi \in \Sigma^{n-1}_{x,\lambda}, \\ v_{x,\lambda}(\eta) &\le v(\eta) \quad \text{for } \eta \in \Sigma^n_{x,\lambda}. \end{aligned}$$

As being shown in the proof of Theorem 3.1, this is impossible. This completes the proof. □

Acknowledgements
M. Zhu would like to thank Y.Y. Li and E. H. Lieb for their interests in the paper and their comments related to Lemma 3.5 and 3.6. He also thanks R. Frank and P. Yang for valuable discussions that lead to the general extension of sharp HLS inequality. This paper is written while J. Dou was visiting the Department of Mathematics, The University of Oklahoma. He thanks the department for its hospitality. The work of J.Dou is partially supported by the National Natural Science Foundation of China (Grant No. 11101319) and CSC project for visiting The University of Oklahoma.

References

[1] J. Ai, K-S. Chou, J. Wei, Self-similar solutions for the anisotropic affine curve shortening problem, Calc. Var. Partial Differential Equations 13 (2000), 311-337.

[2] W. Beckner, Sharp Sobolev inequalities on the sphere and the Moser-Trudinger inequality, Ann. of Math. 138 (1993), 213-242.

[3] H. J. Brascamp; E. H. Lieb; J. M. Luttinger, A general rearrangement inequality for multiple integrals. J. Functional Analysis 17 (1974), 227?37.

[4] H. Brezis, Uniform estimates for solutions of $-\Delta u = V(x)u^p$, in Partial Differential Equations and related subjects, Trento, 1990, (M. Miranda, ed.), Longman, 1992, p. 38-52.

[5] H. Brezis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88 (1983), 486-490.

[6] H. Brezis, T. Kato, Remarks on the Schrödinger operator with singular complex potentials, J. Math. Pures Appl. 58 (1979), 137-151.

[7] X. Cabré, J. Tan, Positive solutions of nonlinear problems involving the square root of the Laplacian, Adv. Math. 224 (2010), 2052-2093.

[8] L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, Comm. Partial. Diff. Equ. 32 (2007), 1245-1260.

[9] S.-Y. Chang, M. Gonzalez, Fractional Laplacian in conformal geometry, Adv. Math. 226 (2011), 1410-1432.

[10] W. Chen, C. Li, Methods on Nonlinear Elliptic Equations, AIMS Ser. Differ. Equ. Dyn. Syst. vol. 4, 2010.

[11] W. Chen, C. Li, Super polyharmonic property of solutions for PDE systems and its applications, arxiv: 1110.2539.

[12] W. Chen, C. Li, B. Ou, Classification of solutions for an integral equation, Comm. Pure Appl. Math. 59 (2006), 330-343.

[13] Y. Chen, L. Wu, Second Order Elliptic Equations and Elliptic Systems, Translations of Mathematical Monographs. AMS, 1998.

[14] J. Dou, M. Zhu, Two dimensional L_p Minkowski problem and nonlinear equations with negative exponents, Adv. Math. 230 (2012), 1209-1221.

[15] J. Dou, M. Zhu, On the extension of Hardy-Littewood-Sobolev inequality II: The negative exponent, in preparation.

[16] J. Dou, M. Zhu, Negative power Hardy-Littewood-Sobolev inequality on the upper half space, in preparation.

[17] J. F. Escobar, Sharp constant in a Sobolev trace inequality, Indiana Univ. Math. J. 37, (1988), 687-698.

[18] J. F. Escobar, The Yamabe problem on manifolds with boundary, J. Differential Geom. 35 (1992), 21-84.

[19] J. F. Escobar, Conformal deformation of a Riemannian metric to a scalar flat metric with constant mean curvature on the boundary, Ann. of Math. 136 (1992), 1-50.

[20] R. L. Frank, E. H. Lieb, Inversion positivity and the sharp Hardy-Littlewood-Sobolev inequality, Calc. Var. Partial Differential Equations 39 (2010), 85-99.

[21] R. L. Frank, E. H. Lieb, Sharp constants in several inequalities on the Heisenberg group, Ann. of Math. to appear.

[22] M. González, R. Mazzeo, Y. Sire, Singular solutions of fractional order conformal Laplacians, J. Geom. Anal. DOI: 10.1007/s12220-011-9217-9.

[23] M. González, J. Qing, Fractional conformal Laplacians and fractional Yamabe problems, arXiv: 1012.0579v1

[24] L. Gross, Logarithmic Sobolev Inequalities, Amer. J. Math. 97 (1976), 1061-1083.

[25] Z. Han, Y. Y. Li, The Yamabe problem on manifolds with boundary: existence and compactness results, Duke Math. J. 99 (1999), 489-542.

[26] F. Hang, P. Yang, The Sobolev inequality for Paneitz operator on three manifolds, Calc. Var. Partial Differential Equations 21 (2004), 57-83.
[27] F. Hang, X. Wang, X. Yan, Sharp integral inequalities for harmonic functions, Comm. Pure Appl. Math. 61 (2008), 0054-0095.
[28] F. Hang, X. Wang, X. Yan, An integral equation in conformal geometry, Ann. Inst. H. Poincaré Analyse Non Linéaire 26 (2009), 1-21.
[29] G. H. Hardy, J. E. Littlewood, Some properties of fractional integrals (1), Math. Zeitschr. 27 (1928), 565-606.
[30] G. H. Hardy, J. E. Littlewood, On certain inequalities connected with the calculus of variations, J. London Math. Soc. 5 (1930), 34-39.
[31] T. Jin, Y. Y. Li, J. Xiong, On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions, arXiv: 1111.1332v1
[32] T. Jin, Y. Y. Li, J. Xiong, On a fractional Nirenberg problem, part II: existence of solutions, in preparation, 2011.
[33] T. Jin, J. Xiong, A fractional Yamabe flow and some applications, arXiv: 1110. 5664v1
[34] Y. Y. Li, Remark on some conformally invariant integral equations: the method of moving spheres, J. Eur. Math. Soc. 6 (2004), 153-180.
[35] Y. Y. Li, L. Zhang, Liouville type theorems and Harnack type inequalities for semilinear elliptic equations, J. D'Anal. Math. 90 (2003), 27-87.
[36] Y. Y. Li, M. Zhu, Uniqueness theorems through the method of moving spheres, Duke Math. J. 80 (1995), 383-417.
[37] Y. Y. Li, M. Zhu, Sharp Sobolev trace inequalities on Riemannian manifolds with boundaries, Comm. Pure Appl. Math. 50 (1997), 449-487.
[38] E. H. Lieb, Existence and Uniqueness of the Minimizing Solution of Choquard's Non-Linear Equation, Studies in Appl. Math. 57, 93-105 (1977).
[39] E. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. of Math. 118 (1983), 349-374.
[40] E. Lieb, M. Loss, Analysis, 2nd ed. Graduate Studies in Mathematics, 14. American Mathematical Society, Providence, R.I. 2001.
[41] Y. Ni, M. Zhu, Steady states for one dimensional curvature flows, Comm. Contemp. Math. 10 (2008), 155-179.
[42] R. O'Neil, Convolution operators and $L(p,q)$ spaces, Duke Math. J. 30 (1963), 129-142.
[43] R. Schoen, S.-T. Yau, Conformally flat manifolds, Kleinian groups and scalar curvature, Inventiones Mathematicae, 92 (1988), 47-71.
[44] S. L. Sobolev, On a theorem of functional analysis, Mat. Sb. (N.S.) 4 (1938), 471-479. A. M. S. Transl. Ser. 2, 34 (1963), 39-68.
[45] E. M. Stein, Singular integrals and differentiability properties of functions, Princeton Mathematical Series, 30. Princeton University Press, Princeton, N.J. 1970.
[46] E. M. Stein, G. Weiss, Fractional integrals in n-dimensional Euclidean space, J. Math. Mech. 7 (1958), 503-514.
[47] X. Wang, M. Zhu, Personal communication, June 2012.
[48] P. Yang, M. Zhu, On the Paneitz energy on standard three sphere, ESAIM Control Optim. Calc. Var. 10 (2004), 211-223.

Jingbo Dou, School of Statistics, Xi'an University of Finance and Economics, Xi'an, Shaanxi, 710100, China and Department of Mathematics, The University of Oklahoma, Norman, OK 73019, USA

E-mail address: `jbdou@xaufe.edu.cn`

Meijun Zhu, Department of Mathematics, The University of Oklahoma, Norman, OK 73019, USA

E-mail address: `mzhu@ou.edu`