# Chapter 8 Balancing and solving equations

**Introduction**

- Solving equations
- Addition invariant (Prop 8.1)
- Multiplication invariant (Prop 8.2)
- Symmetry (Prop 8.3)
- Transitive Property (Prop 8.4)

## 8.1 Setting up the equations for word problems

Can you figure out which number is five more than eight? Yes, you can. Then can you find this number:

*One third of the number is two more than one quarter of a new number which is two less than the original number?*

Feel dizzy because of too many relations? To find the number satisfying certain conditions is a reverse process. It will be harder and harder if more conditions are given. The way to set up an equation, then to solve the equation is a smart move: we can solve the problem reversely step by step (so your brain can take a short break after each step). The difficult part to learn is how to set up the equation.

**Example 8.1** Find a number so that one third of the number is two more than one quarter of a new number which is two less than the original number.

*Solution:* By representing the number by variable $x$, we then transfer English words into mathematical language:

$$\text{One third of the number} \Leftrightarrow \frac{1}{3}x;$$

$$\text{A new number which is two less than the original number} \Leftrightarrow x-2;$$

$$\text{One quarter of new number} \Leftrightarrow \frac{1}{4}\cdot(x-2).$$

We finally write the relation in a mathematical way:

$$\frac{1}{3}x = 2+\frac{1}{4}\cdot(x-2). \tag{8.1}$$

Once we can find the number $x$ satisfying (8.1), we then find the number.

The next game is: how can you solve equations like (8.1). To be continued. □

## 8.2 Solving equations

We have following balance rules for operations.

**Proposition 8.1. Addition invariant**

*If two numbers are the same:* $A = B$, *then*

$$A + C = B + C \tag{8.2}$$

*for any number* $C$.

**Proposition 8.2. Multiplication invariant**

*If two numbers are the same:* $A = B$, *then*

$$A \times C = B \times C \tag{8.3}$$

*for any number* $C$.

The above two rules implies the following (can you verify this?):

**Proposition 8.3. Symmetric property**

*For any two numbers* $A$, $B$,

$$A = B \quad \textit{if and only if} \quad B = A. \tag{8.4}$$

We will use the above rules to solve equations.

**Example 8.2** Find $x$ satisfying

$$5x - 12 = 13.$$

*Solution:* Using Proposition 8.2, we add 12 to both sides, and have

$$5x - 12 + 12 = 13 + 12.$$

Using associative law we can simplify the above as

$$5x = 25.$$

Now using Proposition 8.3, we multiply both sides by $\frac{1}{5}$, and have

$$5x \cdot \frac{1}{5} = 25 \cdot \frac{1}{5}.$$

Again, we simplify the above and have

$$x = 5.$$

□

**Exercise 8.1** Solve the equation

$$\frac{1}{2}x + 3 = 7.$$

**Example 8.3** Set up the equation and solve the word problem: Find a number so that half of the difference between this number and 4 is 3. There shall be two such numbers.

*Solution:* Let $x$ be this number.

If $x$ is greater than 4, then

$$\frac{1}{2}(x-4)=3.$$

Multiplying both sides by 2, we have

$$x-4=6.$$

Adding 4 to both sides, we have $x=10$.

If $x$ is less than 4, then

$$\frac{1}{2}(4-x)=3.$$

Multiplying both sides by 2, we have

$$4-x=6.$$

Adding $x$ to both sides, we have

$$4=6+x.$$

Now, adding $-6$ to both sides, we have $-2=x$, that is: $x=-2$.

Answer: this number is either 10, or $-2$.

□

In dealing with equality, the next rule is often used as well (in fact, all propositions in this section were given by Euclid 2000 years ago in his famous book *Elements*).

**Proposition 8.4. Transitive Property**

*For any three numbers $A$, $B$, and $C$, if*

$$A=B \text{ and } B=C,$$

*then*

$$A=C.$$

♠

To solve a bit more complicated equation like (8.1), we need to simplify algebraic expressions on both sides. We will discuss this in the next chapter.

**My note** **Date:**

## Chapter 8 Exercises

1. Basic skills: Solve the following equations:
   (a). $x - \frac{1}{2} = \frac{1}{4}$
   (b). $2x - 3 = -x + 9$
   (c). $\frac{x-4}{2} = 2x + 7$
   (d). $\frac{1}{2}x + 2 = \frac{1}{4}x + 3$
2. Jack has some money in his pocket. He gives his mom one third of his money and finds that he only have 4 dollar left. How much money does he give to his mom?
3. Ann's age is one-sixth of her dad's age. However, after five years, her age will be two-seventh of her dad's age. What is Ann's age now?
4. Entertaining math: For what $a$, you can solve equation for $x$ variable? What is $x$?
   (a). $ax = 1$
   (b). $ax = x + 1$
5. Entertaining math: For what $a$, the following equation has only one solution for $x$ variable?

$$(a + 1)x = 0.$$