Chapter 10 Polynomials and algebraic operations

Introduction

- Monomials (Definition 10.1)
- Polynomials (Definition 10.2)
- Polynomial operations
- Vertical multiplications and divisions

10.1 Polynomials: Definition

Definition 10.1. Monomial

If n is a natural number, and c is any number, we call the algebraic expression

$$cx^n$$

a monomial, where we usually call number c the coefficient, and n the degree of the monomial. ♣

Definition 10.2. Polynomials

A polynomial is a summation of some monomials. If we arrange the summation according to the degree of each monomial, a polynomial appears in the form of

$$c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0.$$

We then call the largest degree among all monomials the degree of the polynomial. ♣

10.2 Polynomials: Operation

The reason that there are not two terms with the same degree in the above expression is that we can combine them (using the distributive law). For example

$$3x^2 + 5x^2 = (3+5) \cdot x^2 = 8x^2.$$

This remark also indicates that we can apply the addition operation to two polynomials. As long as the degrees of two monomials in the expression are the same, we can add the coefficients together and obtain one term.

Example 10.1

$$3x^2 - x^2 + 10x^2 = (3 - 2 + 10) \cdot x^2 = 11x^2.$$

Example 10.2

$$\begin{aligned}(5x^3 - 4x^2 + 7) + (x^5 + 4x^2 - 9) &= 5x^3 - 4x^2 + 7 + x^5 + 4x^2 - 9 \\ &= x^5 + 5x^3 - 4x^2 + 4x^2 + 7 - 9 \\ &= x^5 + 5x^3 + (-4x^2 + 4x^2) + (7 - 9) \\ &= x^5 + 5x^3 - 2.\end{aligned}$$

From the formula for exponential expression, we can apply multiplication to two monomials:

$$cx^n \times dx^m = c \times d \times x^n \times x^m = cdx^{m+n}.$$

To multiply two polynomials, we need distributive law.

Example 10.3

$$5x^2(2x^3 + 3x) = 5x^2 \cdot (2x^3) + 5x^2 \cdot (3x) = 10x^5 + 15x^3.$$

Example 10.4

$$\begin{aligned}(x+1)(x+1) &= (x+1) \cdot x + (x+1) \cdot 1 \\ &= x \cdot x + 1 \cdot x + x + 1 \\ &= x^2 + x + x + 1 \\ &= x^2 + 2x + 1.\end{aligned}$$

One may wonder whether we can use the vertical form to do the computation. The answer is yes. For this, you may read "deep reading" part in this chapter.

✍ **Exercise 10.1** Find

(1).

$$(x+a)(x+a) =$$

(2).

$$(x-a)(x-a) =$$

(3).

$$(x-a)(x+a) =$$

You will remember the results in the above exercise: they are called special products (or formulas): (1) is called the square of a sum; (2) is called the square of a difference; and (3) is called the difference of squares.

Return to the solution of (8.1). We first simplify both sides:

$$\frac{1}{3}x = 2 + \frac{1}{4}x - \frac{2}{4} = \frac{1}{4}x + 2 - \frac{1}{2} = \frac{1}{4}x + \frac{3}{2}.$$

So

$$\frac{1}{3}x - \frac{1}{4}x = \frac{3}{2}.$$

Combining like terms, we have

$$(\frac{1}{3} - \frac{1}{4})x = \frac{3}{2}.$$

That is

$$\frac{1}{12}x = \frac{3}{2}.$$

Using Proposition 8.3, we have

$$x = \frac{3}{2} \cdot 12 = 18.$$

My note **Date:**

10.3 Deep reading: Polynomial as a base-x number system and the vertical multiplication and division.

Viewing polynomials as base-x numbers, we can again use the vertical form to do the computation.

Example 10.5 Calculate $(3x + 8) \times (4x + 2)$.

Solution:

First step: compute $(3x + 8) \times 2 = 6x + 16$. Note that no carrying number to the usual ten position.

$$\begin{array}{rrr} & 3 & 8 \\ \times & 4 & 2 \\ \hline & 6 & 16 \end{array}$$

(Or to be more precise,

$$\begin{array}{rrr} & 3x & 8 \\ \times & 4x & 2 \\ \hline & 6x & 16 \end{array}$$

if we do not want to omit base x.)

Second step: continue to compute $(3x + 8) \times (4x) = 12x^2 + 32x$. Note the shift of the position.

$$\begin{array}{rrrr} & & 3 & 8 \\ \times & & 4 & 2 \\ \hline & & 6 & 16 \\ & 12 & 32 & \end{array}$$

(Or to be more precise,

$$\begin{array}{rrrr} & & 3x & 8 \\ \times & & 4x & 2 \\ \hline & & 6x & 16 \\ & 12x^2 & 32x & \end{array}$$

if we do not want to omit base x.) Final answer: add numbers together to get the result.

$$\begin{array}{rrrr} & & 3 & 8 \\ \times & & 4 & 2 \\ \hline & & 6 & 16 \\ + & 12 & 32 & \\ \hline & 12 & 38 & 16. \end{array}$$

Now we have to understand what we really get: at constant place (usually, we call this the ones place. Apparently, we can not call it ones place here since we have a two digit number), we have 16; At x place (usually we call this position the tens place if we use base 10 system. Apparently, it is not good to call it tens place since we have a two digit number here again,) we have $38x$; At x^2 place: $12x^2$. So we get the result: $12x^2 + 38x + 16$.

(Or to be more precise,

$$\begin{array}{rrrr} & & 3x & 8 \\ \times & & 4x & 2 \\ \hline & & 6x & 16 \\ + & 12x^2 & 32x & \\ \hline & 12x^2 & 38x & 16 \end{array}$$

if we do not want to omit base x.)

□

Slightly difficult, we can carry out the long division as well.

Example 10.6 Calculate $(x^3+1) \div (x+1)$.

Solution: First step, we try x^2 (since $x^2 \cdot x = x^3$, x^3 is the leading term):

$$\begin{array}{rr|rrrr} & & & x^2 & & \\ \hline x+ & 1\ ) & x^3 & +0x^2 & +0x & +1 \\ & & -(x^3 & +x^2) & & \\ \hline & & & -x^2 & +0x & \end{array}$$

Secondly, we choose $-x$ (since $-x \cdot x = -x^2$, the leading term of the remainder term):

$$\begin{array}{rr|rrrr} & & & x^2 & -x & \\ \hline x+ & 1\ ) & x^3 & +0x^2 & +0x & +1 \\ & & -(x^3 & +x^2) & & \\ \hline & & & -x^2 & +0x & \\ & & & -(-x^2 & -1x) & \\ \hline & & & & x & +1 \\ & & & & -(x & +1) \\ \hline & & & & & 0 \end{array}$$

Finally, we choose 1 (since $1 \cdot x = x$, the leading term of the remainder term):

$$
\begin{array}{r|rrrr}
 & & x^2 & -x & +1 \\
\cline{2-5}
x+\ \ 1 & x^3 & +0x^2 & +0x & +1 \\
 & -(x^3 & +x^2) & & \\
\cline{2-5}
 & & -x^2 & +0x & \\
 & & -(-x^2 & -1x) & \\
\cline{3-5}
 & & & x & +1 \\
 & & & -(x & +1) \\
\cline{4-5}
 & & & & 0
\end{array}
$$

Thus, we have

$$(x^3 + 1) \div (x + 1) = x^2 - x + 1.$$

□

✍ **Exercise 10.2** Calculate $100001 \div 11$.

✍ **Exercise 10.3** Calculate $(x^5 + 1) \div (x + 1)$.

Chapter 10 Exercises

1. Basic skills: Using vertical form to calculate

   (a). 311×211

   (b). $(3x^2 + x + 1)(2x^2 + x + 1)$

   (c). $(50 - 1) \times (50 + 1)$

   (d). $(x - 1)(x + 1)$

2. Basic skills: Calculate

   (a). $2499 \div 49$

   (b). $(x^3 - 1) \div (x - 1)$

   (c). $(x^3 - 3x + 2) \div (x - 1)$

   (d). $(2x^4 + 3x^3 + 6x^2 + 4x + 3) \div (x^2 + x + 1)$

3. Entertaining math. Solve the equation for variable x.

   (a).

$$\frac{1}{x+2} = \frac{4}{x-3}$$

(b).

$$(x-2)(x+4) = (x-1)(x-2)$$

(c).

$$\frac{1-\frac{x}{2}}{2} = \frac{x}{2} - 1$$