Chapter 12 Algebraic operations and factorization

Introduction

- Special products (Prop 12.1)
- Rationalize denominators
- Conjugate numbers (Definition 12.1)
- Factorization
- Rational polynomials

12.1 Some special products

It is nice to summarize certain patterns in doing computation. Here are a few common tricks:

Example 12.1 A number is divisible by 3 if the sum of all its digits is divisible by 3.

Example 12.2 A number is divisible by 9 if the sum of all its digits is divisible by 9.

The reason for these tricks is not very hard to understand (oh, you need to know the *reason* as a parent): for any non-negative integer m, the remainder of 10^m divided by 3 or 9 is always 1. So the remainder of a number divided by 3 or 9 is the sum of all its digits.

There are other tricks. For example,

$$25^2 - 24^2 = 25 + 24 = 49.$$

Why? We will see shortly that it will follow from some special products.

Quite often we will use the following three formulas.

Proposition 12.1. Special products

(1) Square of a sum

$$(a+b)^2 = a^2 + 2ab + b^2.$$

(2) Square of a difference

$$(a-b)^2 = a^2 - 2ab + b^2.$$

(3) Difference of squares

$$(a-b)(a+b) = a^2 - b^2.$$

Students need to grasp these formulas by deriving them numerous times and the fun comes from these computations.

Example 12.3 Compute

$$(a-b)^2(a+b)^2.$$

Solution:

Solution. One way is

Method 1.

$$\begin{aligned}(a-b)^2(a+b)^2 &= (a^2-2ab+b^2)(a^2+2ab+b^2) && \text{(Proposition 12.1 (1) and (2))}\\ &= (a^2+b^2-2ab)(a^2+b^2+2ab)\\ &= (a^2+b^2)^2-(2ab)^2 && \text{(Proposition 12.1 (3))}\\ &= a^4+2a^2b^2+b^4-4a^2b^2 && \text{(Proposition 12.1 (1))}\\ &= a^4-2a^2b^2+b^4.\end{aligned}$$

But the fast way is the following.

Method 2.

$$\begin{aligned}(a-b)^2(a+b)^2 &= [(a-b)(a+b)]^2\\ &= [a^2-b^2]^2 && \text{(Proposition 12.1 (3))}\\ &= a^4-2a^2b^2+b^4. && \text{(Proposition 12.1 (2))}\end{aligned}$$

□

12.2 Rationalizing denominators

The difference of squares formula is often used in rationalizing a fraction with a radical denominator.

Example 12.4 Rationalize the denominators in fraction

$$\frac{1}{\sqrt{3}-\sqrt{2}}, \quad \text{and} \quad \frac{2}{5-\sqrt{7}}.$$

Solution:

$$\begin{aligned}\frac{1}{\sqrt{3}-\sqrt{2}} &= \frac{1}{\sqrt{3}-\sqrt{2}}\cdot\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}}\\ &= \frac{\sqrt{3}+\sqrt{2}}{(\sqrt{3})^2-(\sqrt{2})^2}\\ &= \sqrt{3}+\sqrt{2},\end{aligned}$$

and

$$\begin{aligned}\frac{2}{5-\sqrt{7}} &= \frac{2}{5-\sqrt{7}} \cdot \frac{5+\sqrt{7}}{\sqrt{5}+\sqrt{7}}\\ &= \frac{10+2\sqrt{7}}{(5)^2-(\sqrt{7})^2}\\ &= \frac{10+2\sqrt{7}}{18}\\ &= \frac{5+\sqrt{7}}{9}.\end{aligned}$$

□

From the above example, one may realizes that we can introduce

Definition 12.1. Conjugate-2

For two natural numbers a and b, we say $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ are conjugate to each other. ♣

12.3 Tricks for computing products

Reversely, factorizing a number or a polynomial will help us to understand and sometimes simplify an expression.

Example 12.5 Find

$$25^2 - 24^2.$$

Everyone may know that

$$25^2 - 24^2 = 625 - 486 = 49.$$

Here is a way for doing it mentally:

$$25^2 - 24^2 = (25-24)(25+24) = 49.$$

Example 12.6 There is a trick to mentally compute square of a number ended with 5, say, for example, 35^2. We realize that

$$\begin{aligned}35^2 - 5^2 &= (35-5)\times(35+5)\\ &= 30\times 40.\end{aligned}$$

So

$$35^2 = 30\times 40 + 5^2 = 1200 + 25 = 1225.$$

Generally, for any number a between 1 and 10, we have

$$(10a+5)^2 - 5^2 = (10a+5-5)\times(10a+5+5)$$
$$= (10a)\times[10(a+1)].$$

Thus

$$(10a+5)^2 = 10a\times 10(a+1) + 5^2 = a\times(a+1)\times 100 + 25.$$

Exercise 12.1 Mentally compute $15^2,\ 25^2,\ 55^2,\ 85^2$.

12.4 Factorization and solving high order equations

Factorization helps us to solve more complicated equations.

Example 12.7 Solve

$$(x-2)(x-3) = 0.$$

Note *Logically (can you prove this?), from*

$$A\times B = 0$$

we can obtain

$$\textit{either}\ \ A = 0\ \ \textit{or}\ \ B = 0.$$

Cautious: this doe not rule out the possibility that both A and B are zero.

So we have

Solution:

$$\text{either}\ \ x-2=0\ \ \text{or}\ \ x-3=0.$$

That is

$$\text{either}\ \ x=2\ \ \text{or}\ \ x=3.$$

□

Now suppose that we are asked to solve

$$x^2-5x+6=0.$$

we argue that if we know

$$x^2-5x+6=(x-2)(x-3),$$

then we can break this equation involving second degree polynomial (we shall call it second order equation later) into two equations involving first degree polynomials (we naturally call it first degree equation, and quite often, call it linear equation). Solving these two equations we have

$$\text{either}\ \ x=2\ \ \text{or}\ \ x=3.$$

So, to find two numbers x_1 and x_2 satisfying

$$x^2 + ax + b = 0$$

is equivalent to factorize $x^2 + ax + b$ into form

$$(x - x_1)(x - x_2).$$

The above strategy may fail for the following exercise.

✍ **Exercise 12.2** Solve equation

$$(x - 1)(x - 3) = 3.$$

It turns out factorization is difficult for most students. Here is the general strategy to factorize a polynomial:

i) Find common factors and pull it out.

ii) Regroup and find certain common factors.

iii) Use standard formula.

Most important: students need to do a lot of exercises. Lots of!

Example 12.8 Factorize

1).

$$x^2 - 3x.$$

2).

$$x^3 - 3x^2 - x + 3.$$

Solution:

1).

$$x^2 - 3x = x(x - 3).$$

2).

$$\begin{aligned} x^3 - 3x^2 - x + 3 &= (x^3 - 3x^2) - (x - 3) \\ &= x^2(x - 3) - (x - 3) \\ &= (x^2 - 1)(x - 3) \\ &= (x + 1)(x - 1)(x - 3). \end{aligned}$$

□

Conversely, we see that one can factorize

$$x^2 + (b + a)x + ab = (x + a)(x + b).$$

This can also be described by the cross product diagram:

$$\begin{array}{ccc} 1 & & a \\ & \nwarrow \ \nearrow & \\ & \swarrow \ \searrow & \\ 1 & & b \end{array}$$

For a given quadratic form

$$x^2 + px + q,$$

we need to factorize $q = a \cdot b$ so that $a + b = p$, then

$$x^2 + px + q = (x + a)(x + b).$$

✍ **Exercise 12.3** Factorize

1).

$$x^2 + 7x + 12;$$

2).

$$x^2 - 7x + 12;$$

3).

$$2x^2 + 7x + 5;$$

4).

$$2x^2 - 7x + 5.$$

5).

$$x^3 - 3x^2 + 2.$$

Hint for 5): need to regroup.

12.5 Rational polynomials

The expression

$$\frac{\text{One polynomial}}{\text{Another polynomial}}$$

is called a rational polynomial. For example

$$\frac{x^3 + 2x^2 - 4}{x - 3}, \quad \frac{x^3}{x^2 - 3x + 5}$$

are rational polynomials. And

$$x^{-3} \cdot (x^2 - 2x + 4)$$

is also a rational polynomial, since

$$x^{-3} \cdot (x^2 - 2x + 4) = \frac{x^2 - 2x + 4}{x^3}.$$

If the denominator of a rational polynomial is a monomial, one can change the denominator to numerator by using a negative power, and then to combining with like terms.

Example 12.9 Simplify

$$\frac{x^2 - x + 5}{x^2} - 1 + x^{-1} - 4x^{-2}.$$

Solution:

$$\begin{aligned}\frac{x^2 - x + 5}{x^2} - 1 + x^{-1} - 4x^{-2} &= (x^2 - x + 5) \cdot x^2 - 1 + x^{-1} - 4x^{-2} \\ &= 1 - x^{-1} + 5x^{-2} - 1 + x^{-1} - 4x^{-2} \\ &= x^{-2}.\end{aligned}$$

□

Example 12.10 Simplify

$$\frac{x^2 - 2x}{2 - x}.$$

Solution:

$$\begin{aligned}\frac{x^2 - 2x}{2 - x} &= \frac{x(x-2)}{-(x-2)} \\ &= \frac{x}{-1} \\ &= -x.\end{aligned}$$

□

✍ **Exercise 12.4** Simplify

1).

$$\frac{x^2 - 3x + 5}{x^{-2}} + 3x^3 - 3x^2 - x^4.$$

2).

$$\frac{x^2 - 4x + 3}{3 - x}.$$

Often, students like to ask why we need to simplify an expression. Here is one exercise that may help you to answer the question partially:

✍ **Exercise 12.5** Evaluate

$$\frac{x^2 - x + 5}{x^2} - 1 + x^{-1} - 4x^{-2}$$

for $x = 0.2$.

My note **Date:**

Chapter 12 Exercises

1. Basic skills: Calculate
   (a). $(x-2)(x+3)$.
   (b). $(x^2-2)(x^2+3)$
   (c). $(x-1)(x+1)(x^4+x^2+1)$
   (d). $(x+1)(x^4-x^3+x^2-x+1)$
2. Basic skills: Rationalize denominators
   (a). $\frac{3}{\sqrt{6}}$
   (b). $\frac{1-2\sqrt{3}}{\sqrt{12}}$
   (c). $\frac{1}{\sqrt{x+1}+\sqrt{x-1}}$
   (d). $\frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}+\sqrt{3}}$
3. Basic skills: Factorize (in real number system):
   (a). x^2+4x+3
   (b). x^4+4x^2+3
   (c). $x^3-13x+12$
   (d). x^6-13x^2+12
4. Entertaining math.
   (a). Calculate: $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\cdots+\frac{1}{2019\times 2020}$
   (b). Calculate: $\frac{2}{\sqrt{3}+1}+\frac{2}{\sqrt{5}+\sqrt{3}}+\cdots+\frac{2}{\sqrt{121}+\sqrt{119}}$
5. Challenging math.
   (a). Factorize:
   $$x^4+x^2+1$$

(b). Is number $19^4 + 19^2 + 1$ a prime number? Why?

(c). Calculate:

$$\frac{2}{1+1+1} + \frac{4}{2^4+2^2+1} + \cdots + \frac{2k}{k^4+k^2+1} + \cdots + \frac{40}{20^4+20^2+1}$$