Barbers, Painters, and Berry

Famous Paradoxes and their Relation to Mathematics

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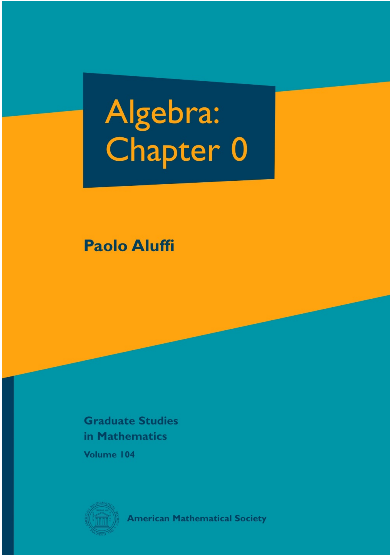
October

Motivation



1.1 Locate a discussion of Russell's paradox, and understand it.

What is a Paradox?

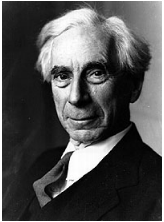
paradox
par · a · dox [par-uh-doks]
noun
1. a statement or proposition that seems self-contradictory or absurd but in reality expresses a possible truth.

http://dictionary.reference.com/browse/paradox?s=t

Russell's Paradox

Bertrand Russell

(1872- 1970)



http://fair-use.org/bertrand-russell/the-principles-of-mathematics/

Principles of Mathematics (1903)

"Pure Mathematics is the class of all propositions of the form p implies q, where p and q are propositions containing one or more variables, the same in the two propositions, and neither p nor q contains any constants except logical constants. And logical constants are all notions definable in terms of the following: Implication, the relation of a term to a class of which it is a member, the notion of such that, the notion of relation, and such further notions as may be involved in the general notion of propositions of the above form."

"Applied mathematics is defined by the occurrence of constants which are not logical."

Significance

- Russell showed inconsistencies with Naive Set Theory

- Motivated work in Foundations and Logic

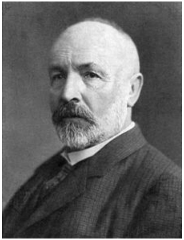
Who Shaves the Barber?

Imagine in the town of Davis there is only one barber and that the barber is male. Addtionally, all the male residents of Davis are required to be clean shaven. A man has two options to be shaven:

- Shave himself
- Go to the barber

Who shaves the barber?

Naive Set Theory



Set Theory first developed as Naive Set Theory by Greg Cantor at the end of the 19th century.

An Incredibly Fast Overview of Set Theory

A set is a collection of objects, symbols, or numbers.

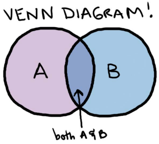
Examples:

- $\{banana, trombone, HarryPotter, pants\}$ is a set of objects.
- $A = \{150, 125, 128, 135, 115, 147\}$ is a set of numbers. I am calling A this set, for convenience sake.
- $B = \{2, 4, 6, 8, 10, 12, ...\} = \{\text{all positive even numbers}\} = \{2k \text{ such that } k \text{ is a positive number}\} = \{2k | k \in \mathbb{N}\}$

An Incredibly Fast Overview of Set Theory

Again, $A = \{150, 125, 128, 135, 115, 147\}$ and $B = \{2k | k \in \mathbb{N}\}$
We can do things with sets, such as:

- Combine them. If we combine A and B we get back A. We call the combining of sets a *union*.
- Intersect them. If we intersect A and B we get only the numbers that are in both sets. In this case, the intersection of A and B are just the even numbers of B, which is $\{128, 150\}$.
- Put them inside of other sets. For example, let's call $C = \{A, B\}$, which means it is the set consisting of the set A and B. (Note: this is *not* the same as the set of the union of A and B)



Stating the Problem in Set Notation

$$R = \{X | X \notin X\}$$

Another Example

List of UCD Math Professors	List of Cities	List of lists that start with the letter L	List of Ice Cream Flavors	List of lists that don't contain themselves
Biello	Dixon	List of UCD Math Professors	Vanilla	List of UCD Math Professors
De Loera	Davis	List of Cities	Chocolate	List of Cities
Gravner	Woodland	List of lists that start with the letter L	Mint Chocolate Chip	List of Ice Cream Flavors
Hunter	Sacramento	List of Ice Cream Flavors	Coffee	List of lists that don't contain themselves?
Kuperberg	Vacaville	List of lists that don't contain themselves	Strawberry	

How to Solve the Issue

Axiomatic Set Theory

1. Extensionality: If two sets have the same elements, they are the same set.

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7. Power Set: For any set X, there is a set, $P(X)$, whose elements are the subsets of X.

The Axioms, cont.

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9. Foundation: If the set X is nonempty, then for some element $A \subseteq X$, A is not equal to X.
10. Well-Ordering Theorem: Any set X can have an ordering such that every nonempty subset of X has a least element.

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If we take a pre-existing set X and a property ϕ we can create a new set.
Suppose that $R = \{X | X \notin X\}$ then $R \in R$ is impossible because it would have to satisfy $R \in X$ and $R \notin R$, but if $R \notin R$ this means $R \notin X$. We can not create a new set, and the axiom can not hold. Contradiction.

Paradox 2

Gabriel's Horn and the Painter's Paradox

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Could we have a 3-dimensional object with finite volume, but infinite surface area?

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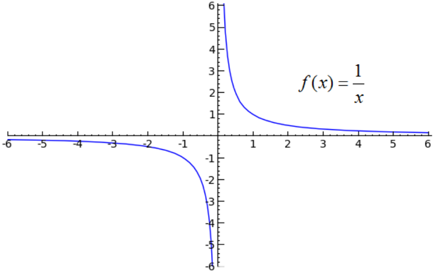
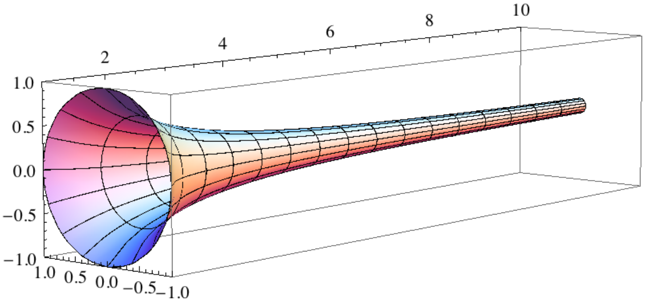
Consider:

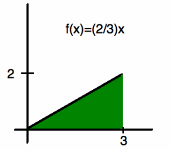


Creating the Horn



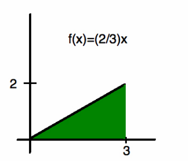
http://demonstrations.wolfram.com/GabrielsHorn/

Solids of Revolution



First we rotate around x-axis

Solids of Revolution



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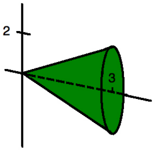

Then we cut up into tiny slices

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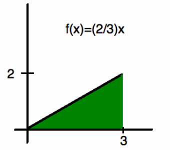

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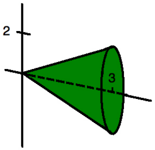

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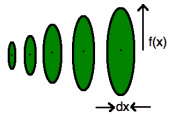

Each slice will have radius is $f(x)$

Height will be dx (the vanishingly smaller Δx)

Calculating the Volume

Volume individually is: $V = \pi r^2 h$
Thus the volume of each disk is $dV = \pi[f(x)]^2 dx$
Total volume: $V = \int_a^b \pi[f(x)]^2 dx$

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$$
\begin{aligned}
V &= lim_{a\to\infty} \int_1^a \pi \frac{1}{x^2} dx \\
&= lim_{a\to\infty} \pi \left(1 - \frac{1}{a}\right) \\
&= \pi
\end{aligned}
$$

Calculating the Surface Area

Area of each slice: $dA = 2\pi r ds$ Here ds refers to the arc length

Arc length: $ds = \sqrt{[dx]^2 + [dy]^2}$

Surface Area: $A = \int_a^b 2\pi f(x)\sqrt{1 + \left(\frac{dx}{dy}\right)^2}dx$

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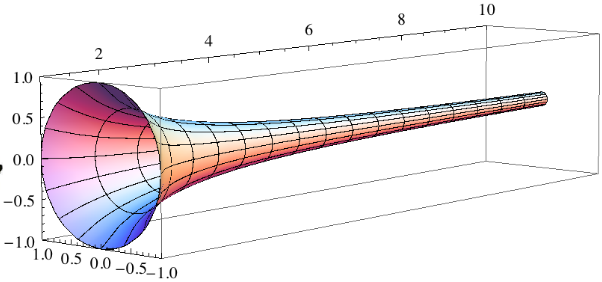
$$\begin{aligned} A &= lim_{a\to\infty} \int_1^a 2\pi \frac{1}{x}\sqrt{1+\frac{1}{x^4}}dx \\ &\geq lim_{a\to\infty} \int_1^a 2\pi \frac{1}{x} dx \\ &\geq lim_{a\to\infty} 2ln(a) \\ &\geq \infty \end{aligned}$$

Painter's Paradox



Can fill inside with finite amount of paint, but need infinite amount of paint for exterior

However...

What if we let paint bleed through?

Thinning Paint

If the thickness of the paint coat becomes vanishingly small, then we could, in theory, paint an infinite surface area with finite paint.

Paradox 3

Berry Paradox: (also discovered by Russell)

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"The least integer not nameable in fewer than nineteen syllables"
(this CAN have a least, by the Well-Ordering Principle)
is itself a name consisting of eighteen syllables; hence the least integer not nameable in fewer than nineteen syllables can be named in eighteen syllables, which is a contradiction.

The End

Thank you!!