# Equivalence of Categories

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September 24, 2015

Recall, the following definition:

**Definition**: (covariant) functor
A **(covariant) functor** $F : C \to D$ between categories $C$ and $D$ consists of

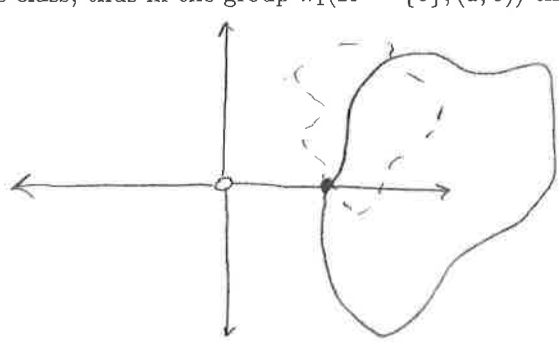
- an object $F(c) \in D$ for all $c \in C$
- a morphism $F(f) : F(c) \to F(c') \in D$ for all $f : c \to c' \in C$
- satisfies the *functorality axioms:*
  - for any composable pair $f, g \in C$, $F(g) \circ F(f) = F(g \circ f)$
  - for each $c \in C$, $F(1_c) = 1_{F(c)}$

Here I use the word "covariant" because there is actually more than one type of functor between categories. I define the other type, the *contravariant functor*, after I give one more explicit example of a covariant functor.

Both of Gordon Brown's talks introduced many examples of covariant functors. I introduce one more essential to the study of algebraic topology, the $\pi_1$ *functor.*

**Example:** the $\pi_1$ functor
$\pi_1$ is a functor between the category *Top*$^*$ and *Group*, $\pi_1 : Top^* \to Group$. We've seen *Group* before: the objects are groups and the morphisms are group homomorphisms. The category *Top*$^*$ has topological spaces with specified basepoints[1] as objects and basepoint preserving continuous functions as morphisms, i.e. $\pi_1$ sends a continuous function $f : (X, x_0) \to (Y, y_0)$ to a group homomorphism $\pi_1(f)^2 = \pi(X, x_0) \to \pi_1(Y, y_0)$. The group that $\pi_1$ sends $(X, x_0)$, i.e. $\pi_1(X, x_0)$ is called *the fundamental group of X*. Note that this definition is suggestive that the choice of basepoint may not matter. I may sometimes omit the basepoint.

Intuitively, we can think of the fundamental group as the equivalence classes of loops that start and end at the specified basepoint. Two loops are equivalent if we can "deform" one loop to look like the other without exiting the space. Consider the space $\mathbb{R}^2 - \{0\}$ (the $xy$-plane, omitting the origin). The loops drawn below are in the same equivalence class, thus in the group $\pi_1(\mathbb{R}^2 - \{0\}, (1,0))$ they are the same element.

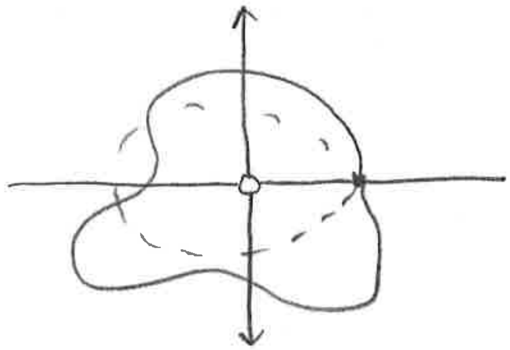


However, the loops that go around the origin, pictured below, are not equivalent to the loops in the first picture

[1]For example, an object is in *Top*$^*$ is not just $\mathbb{R}$, it has a point specified with it, e.g. $(\mathbb{R}, 0)$. An example of a basepoint preserving morphism is $f : (\mathbb{R}, 0) \to (\mathbb{R}, 1)$ where $f(x) = 3x + 1$

[2]Typically the notation to denote $\pi_1(f)$ is $f_*$

since to "deform" loops in the first picture to look like loops in the second picture we must pass through the origin.



On the other hand, when considering the object, $(\mathbb{R}, (1,0))$ in *Top**, all of these loops are equivalent, since the origin is now part of the space. Thus we have the following fact:

**Fact:** $\pi_1(\mathbb{R}, (1,0)) \cong \pi_1(D^2, (1,0))^3 \cong \{0\}$, the group with one element.

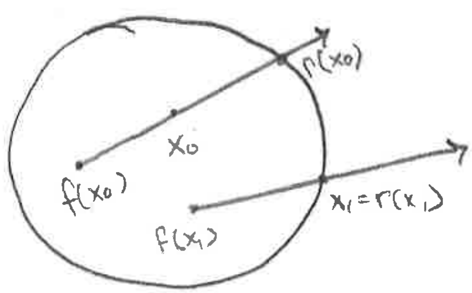
It is beyond the scope of this talk to prove the following fact, so the below is stated without any convincing proof. If you're skeptical, please see *Munkres: Topology, 2nd edition, p.345.*

**Fact:** $\pi_1(\mathbb{R}^2, (1,0)) \cong \pi_1(S^1, (1,0))^4 \cong \mathbb{Z}$.

Back to general functors. The *functorality axioms* can be used to prove an important theorem in Topology, *Brouwer's Fixed Point Theorem.*

**Theorem:** Brouwer's Fixed Point Theorem
Any continuous endomorphism[5] of a 2-dimensional disk has a fixed point.

*Proof:* Suppose $f : D^2 \to D^2$ is an endomorphism such that there are no fixed points, i.e. $f(x) \neq x$ for all $x \in D^2$. Define $r : D^2 \to S^1$ in the following way: make a ray starting at $f(x)$ and passing through $x$. Let $r(x)$ be the point on the ray that intersects the boundary, $S^1$. I've shown an example below.




$r$ is continuous[6] and fixes $S^1$. Call $i : S^1 \to D^2$ the inclusion map and $r \circ i = \mathrm{id}_{S^1}$. Note that

$$\pi_1(r \circ i) = \pi_1(\mathrm{id}_{S^1}) = \mathrm{id}_{\pi_1(S^1)} = \mathrm{id}_{\mathbb{Z}} \qquad \text{using the 2nd functorality axiom}$$

On the other hand, $\pi_1(r) \circ \pi_1(i)$ is a map:

$$\pi_1(S^1) \xrightarrow{i} \pi_1(D^2) \xrightarrow{r} \pi_1(S^1)$$

Since $\pi_1(D^2) \cong \{0\}$, this gives that $\pi_1(r) \circ \pi_1(i) = 0$. By the first functorality axiom, $\pi_1(r \circ i) = \pi_1(r) \circ \pi_1(i)$. Now we've received our contradiction. The idenity map on $\mathbb{Z}$ is certainly not the same as the constant map on

[3] Recall that $D^2$ is the 2-dimensional disk
[4] Recall, $S^1$ is the circle centered at the origin
[5] In *Top** this is a continuous function from a space to itself
[6] Proof of continuity is messy and is not useful for this talk so is left out

$\mathbb{Z}$. ■

As promised, here's the other type of functor:

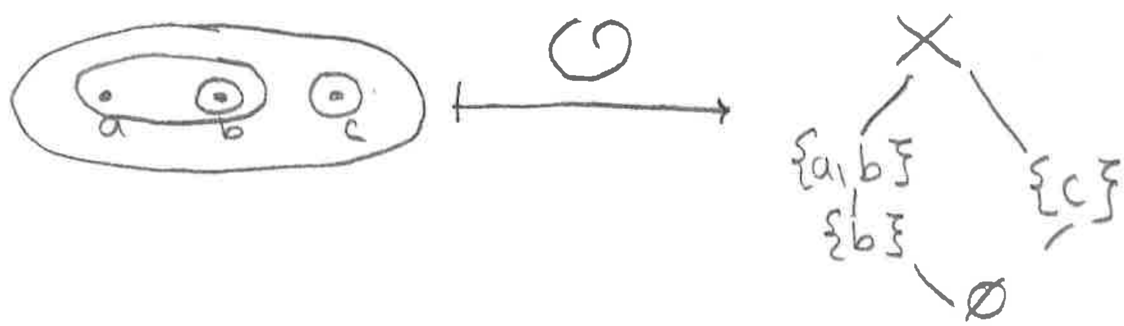
**Definition:** contravariant functor
A **contravariant functor** $F : C \to D$ is a functor consisting of

- an object $F(c) \in D$, for all $c \in C$
- a morphism $F(f) : F(c') \to F(c) \in D$ for all morphisms $f : c \to c'$
- satisfies the *functorality axioms*
  - for any composable pair $f, g \in C$, $F(f) \circ F(g) = F(g \circ f)$
  - for all $c \in C$, $F(1_c) = 1_{F(c)}$

This definition varies from the covariant functor definition in the second and fourth bullet points. Instead of mapping a morphism $f : c \to c'$ to a morphism from $F(c)$ to $F(c')$, it maps it to a morphism from $F(c')$ to $F(c)$. With this comes that the ordering of bullet point four must be flipped as well.

To continue with the Topology theme of the talk I give an example of a contravariant functor, of which I simply call $\mathcal{O}$.

**Example:** the $\mathcal{O}$ functor
The functor $\mathcal{O} : \mathit{Top} \to \mathit{Poset}$ carries a topological space $X$ to its open sets. It takes a continuous function $f : X \to Y$ to $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ of which carries an open $U \subset Y$ to its preimage $f^{-1}(U)$ in $X$ (also open by continuity of $f$). A picture of $\mathcal{O}$ sending an object to an object is pictured below.




In Group Theory there is a notion of two groups being isomorphic if there is a isomorphism between them. In Topology two topological spaces are homeomorphic if there exists a homeomorphism between them. Now that we have functors, we can think of them as morphisms between categories. Is there an similar type of functor that gives some sort of isomorphism of categories?

To answer this question, I need to next introduce *natural transformations.* I've been informed that Paul Plummer is going to be talking about natural transformations next week, so in order not to spoil his talk, I'll try to keep it brief.

**Definition** natural transformation
Given categories $C$ and $D$ and functors $F, G : C \to D$, a **natural transformation** $\alpha : F \to G$ consists of

- a morphism $\alpha_c : F(c) \to G(c)$ in $D$ for each object $c \in C$ so that for any morphism $f : c \to c'$, the following diagram commutes:

$$\begin{tikzcd} F(c) \arrow[r, "\alpha_c"] \arrow[d, "F(f)"] & G(c) \arrow[d, "G(f)"] \\ F(c') \arrow[r, "\alpha_{c'}"] & G(c') \end{tikzcd}$$

**Definition:** natural isomorphism

A **natural isomorphism** is a natural transformation in which every $\alpha$ is an isomorphism.

**Example:** The functors $\mathcal{O}$ and $\mathcal{C}$ are naturally isomorphic.

**Example 1:** A natural isomorphism between $F, G : C \to D$ where $C$ and $D$ are groups regarded as one-object categories is an isomorphism of groups.

Recall,

**Definition:** isomorphism

An **isomorphism** in a category is a morphism $f : X \to Y$ for which there exits a morphism $g : Y \to X$ so that $gf = 1_X$ and $fg = 1_Y$.

**Definition:** equivalence of categories

An **equivalence of categories** consists of functors $F : C \to D$, $G : D \to C$ together with a natural isomorphism $\eta : 1_C \cong G \circ F$ and $\epsilon : F \circ G \cong 1_D$.

**Definition:** equivalent

Categories $C$ and $D$ are **equivalent**, $C \cong D$, if there exists an equivalence between them.

**Example:** Consider a category $C$ with one object $c$ and a single morphism $1_c$ and a category $D$ with two objects $d_1$, $d_2$ with four morphisms, $1_{d_1}$, $1_{d_2}$, and two isomorphisms $\alpha : d_1 \to d_2$ and $\beta : d_2 \to d_1$. The categories $C$ and $D$ are equivalent, we can have $F$ map $c$ to $d_1$, $1_c$ to $1_{d_1}$ and $G$ map $d_1$ to $c$ and $d_2$ to $c$ and all morphisms to $1_c$.

**Nonexample:** If $C$ is the category above and $D$ is the category above without the morphisms $\alpha$ and $\beta$, then $C$ and $D$ are no longer isomorphic.

Before we introduce more equivalent categories we need a few more definitions:

**Definition:** full

A functor $F : C \to D$ is **full** if for each $x$, $y \in C$,

$$\mathrm{Hom}_C(x, y) \to \mathrm{Hom}_D(F(x), F(y))$$

is surjective.

**Definition:** faithful

A functor $F : C \to D$ is **faithful** if for each $x$, $y \in C$,

$$\mathrm{Hom}_C(x, y) \to \mathrm{Hom}_D(F(x), F(y))$$

is injective.

**Definition:** essentially surjective on objects

A functor $F : C \to D$ is **essentially surjective on objects** if for every element $d \in D$ there is some $c \in C$ such that $d$ is isomorphic to $F(c)$.

**Example:** The forgetful functor $U : Grp \to Set$ is faithful since each group maps to a unique set and the group homomorphism maps to a unique set map. $U$ is not full since not every set map is a group homomorphism.

**Example:** The example of the equivalence of categories above is essentially surjective, full, and faithful.

Note, sometimes we call a functor that is both full and faithful as *fully faithful*

**Theorem:**
A functor defining an equivalence of categories is full, faithful, and essentially surjective on objects. The converse is true with the Axiom of Choice.

**Definition:** connected
A category is **connected** if any pair of objects can be "connected" by a zig-zag of morphisms.

**Corollary 1:** Any connected category where every morphism is an isomorphism is equivalent as a category to the automorphism group of any of its objects.

**Fact:** For any topological space $X$, $\pi_1(X)$ is a category where every morphism is basepoint preserving paths up to homotopy equivalence in $X$ and objects are points in $X$, called the *fundamental groupoid.* All morphisms are isomorphisms.

**Corollary:** In a path-connected topological space $X$, any choice of basepoint $x \in X$ yields an isomorphic fundamental group $\pi_1(X, x)$.

*Proof:* Let $x \in X$. The group of automorphisms on $x \in \pi_1(X)$ is exactly $\pi_1(X, x)$. From Corollary 1, every automorphism group is equivalent to $\pi_1(X)$. Thus for any $x$, $\pi_1(X)$ is equivalent to $\pi_1(X, x)$. From Example 1, a pair of equivalences of one-object categories where the objects are regarded as groups is a group isomorphism. Thus $\pi_1(X, x) \cong \pi_1(X, y)$ for all choice of basepoints $x, y \in X$.

**Corollary:** The category $\pi_1(X)$ is equivalent as a category to $\pi_1(X, x_0)$ recognized as a category.

Sources:

"Equivalence of Categories." Wikipedia. Wikimedia Foundation, n.d. Web. 23 Sept. 2015.

Riehl, Emily. "Category Theory in Context." (n.d.): n. pag. 22 Sept. 2015. Web. 22 Sept. 2015.