Name:______________________________

Math 4163 Section 002 | *Practice Exam 3* | *February 17, 2015*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit!

1. Given the following Sturm-Liouville problem

$$\begin{cases} \Delta v + \lambda v = 0 \\ v|_{\partial\Omega} = 0 \end{cases}$$

Show that the eigenfunctions are orthogonal, i.e. show that for two different eigenvalues $\lambda_1 \neq \lambda_2$,

$$\int\int_\Omega \varphi_1(x,y)\varphi_2(x,y)\ dA = 0$$

HINT: You will need Lagrange's identity, $\displaystyle\int\int_\Omega (u\Delta v - v\Delta u)\ dA = \int_{\partial\Omega} (u\nabla v - v\nabla u)\cdot \mathbf{n}\ ds$ and consider for the two Sturm-Liouville problems for λ_1 associated to the solution $\varphi_1(x,y)$ and λ_2 associated to the solution $\varphi_2(x,y)$.

2. Given the following Sturm-Liouville problem

$$\begin{cases} \Delta v + \lambda v = 0 \\ v|_{\partial\Omega} = 0 \end{cases}$$

Show that the eigenvalues are real, i.e. show that $\lambda = \bar{\lambda}$. HINT: You will need Lagrange's identity, $\displaystyle\int\int_\Omega (u\Delta v - v\Delta u)\ dA = \int_{\partial\Omega} (u\nabla v - v\nabla u)\cdot \mathbf{n}\ ds$ and consider for the two Sturm-Liouville problems for λ associated to the solution $\varphi(x,y)$ and $\bar{\lambda}$ associated to the solution $\bar{\varphi}(x,y)$.

3. Given the following ODE:

$$(x^6 + x^2)\frac{d^2y}{dx^2} + (x^7 + x)\frac{dy}{dx} + (6x^5 - 4)y = 0$$

What is the expected approximate behavior of the solutions near $x = 0$.

4. Given the following ODE:

$$(x^8 + x^2)\frac{d^2y}{dx^2} + \left(x^5 + \frac{3}{16}\right)y = 0$$

What is the expected approximate behavior of the solutions near $x = 0$.

5. Using the eigenfunction expansion method, solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} u_t = ku_{xx} + Q(x,t) \\ u_x(0,t) = A(t), \ u(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$$

HINT: You will need Lagrange's identity, $\displaystyle\int_0^L \left(c\frac{d^2h}{dx^2} - h\frac{d^2c}{dx^2} \right) \, dx = \left(c\frac{dh}{dx} - h\frac{dc}{dx} \right)\Big|_0^L .$

6. Using the eigenfunction expansion method, solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} u_t = ku_{xx} + Q(x,t) \\ u_x(0,t) = A(t), \ u_x(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$$

HINT: You will need Lagrange's identity, $\displaystyle\int_0^L \left(c\frac{d^2h}{dx^2} - h\frac{d^2c}{dx^2} \right) \, dx = \left(c\frac{dh}{dx} - h\frac{dc}{dx} \right)\Big|_0^L .$

7. Using the Fourier transform, solve the following PDE subject to the following initial condition on the unbounded domain:

$$\begin{cases} u_t = ku_{xx} + q(x,t) \\ u(x,0) = f(x) \ , \ x \in \mathbb{R} \end{cases}$$

HINT: After you solved the first order ODE, use the fact that $e^a e^b = e^{a+b}$ and when computing the inverse Fourier transform use the fact that

$$\mathcal{F}^{-1}\left[\int F(\omega,s)\ ds\right] = \int \mathcal{F}^{-1}\left[F(\omega,s)\right]\ ds$$

8. Using the Fourier transform, solve the following PDE subject to the following initial condition on the unbounded domain:

$$\begin{cases} u_t = ku_{xxx} \\ u(x,0) = f(x) \ , \ x \in \mathbb{R} \end{cases}$$

HINT: You will need the definition of the Airy function, $Ai(x)$:

$$Ai(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\frac{\omega^3}{3}} e^{-i\omega x}\ d\omega = \frac{1}{\pi}\int_0^\infty \cos\left(\frac{\omega^3}{3} + x\omega\right)\ d\omega$$

9. Let $\mathbf{b}$ be a vector in $\mathbb{R}^2$, that is $\mathbf{b} = (b_1, b_2)$. Also let g be a differentiable function in $\mathbb{R}^2$. Define

$$u(x, y, t) = g(x - b_1 t, y - b_2 t) + \int_0^t f(x + (s - t)b_1, y + (s - t)b_2, s)\ ds$$

Show that $u_t + \mathbf{b} \cdot \nabla u = f$ HINT: Just compute the derivatives and see what happens.