Name:______________________________

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit!

1. Solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} au_{tt} = bu_{xx} - cu_t \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \text{ and } u_t(x,0) = g(x) \end{cases}$$

where a, b, and c are constants and $c^2 < \dfrac{4\pi^2 ab}{L^2}$. You do not need to derive what the coefficients are in the resulting series solution, just state the formulas for them if you can.

2. Solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} au_{tt} = bu_{xx} - cu_t \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \text{ and } u_t(x,0) = g(x) \end{cases}$$

where a, b, and c are constants and $c^2 > \dfrac{n^2\pi^2 ab}{L^2}$ for all n. You do not need to derive what the coefficients are in the resulting series solution, just state the formulas for them if you can.

3. Solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} a(x)b(x)\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(c(x)\dfrac{\partial u}{\partial x}\right) + \alpha(x)u \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

What happens to $u(x,t)$ as $t \to \infty$? You do not need to derive what the coefficients are in the resulting series solution, just state the formulas for them if you can. Also don't forget to state that you are using the Sturm-Liouville Theorem when solving the eigenvalue problem.

4. Solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} a(x)b(x)\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(c(x)\dfrac{\partial u}{\partial x}\right) + \alpha(x)u \\ u(-L,t) = u(L,t) \text{ and } u_x(-L,t) = u_x(L,t) \\ u(x,0) = f(x) \end{cases}$$

What happens to $u(x,t)$ as $t \to \infty$? You do not need to derive what the coefficients are in the resulting series solution, just state the formulas for them if you can. Also don't forget to state that you are using the Sturm-Liouville Theorem when solving the eigenvalue problem.

5. Solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} u_t = k\Delta u \\ u(0,y,t) = u(L,y,t) = 0 \\ u(x,0,t) = u(x,H,t) = 0 \\ u(x,y,0) = \alpha(x,y) \end{cases}$$

where $k > 0$ is a constant. You do not need to derive what the coefficients are in the resulting series solution, just state the formulas for them if you can.

6. Solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} u_t = k\Delta u \\ u_x(0,y,t) = u_x(L,y,t) = 0 \\ u(x,0,t) = u(x,H,t) = 0 \\ u(x,y,0) = \alpha(x,y) \end{cases}$$

where $k > 0$ is a constant. You do not need to derive what the coefficients are in the resulting series solution, just state the formulas for them if you can.

7. Given the following Sturm-Liouville problem

$$\begin{cases} L(h) + \lambda\sigma(x)h = 0 \\ h(a) = h(b) = 0 \end{cases}$$

with

$$L(h) = \frac{d}{dx}\left(p(x)\frac{dh}{dx}\right) + q(x)h$$

Show that the eigenfunctions are orthogonal, i.e. show that for $m \neq n$,

$$\int_a^b \varphi_n(x)\varphi_m(x)\sigma(x)dx = 0$$

HINT: You will need Lagrange's identity, $\int_a^b (uL(v) - vL(u))dx = p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b$ and consider for $m \neq n$ the two Sturm-Liouville problems for λ_n associated to the solution $\varphi_n(x)$ and λ_m associated to the solution $\varphi_m(x)$.

8. Given the following Sturm-Liouville problem

$$\begin{cases} L(h) + \lambda\sigma(x)h = 0 \\ h(a) = h(b) = 0 \end{cases}$$

with

$$L(h) = \frac{d}{dx}\left(p(x)\frac{dh}{dx}\right) + q(x)h$$

Show that the eigenvalues are real, i.e. show that $\lambda = \bar{\lambda}$. HINT: You will need Lagrange's identity, $\int_a^b (uL(v) - vL(u))dx = p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b$ and consider the two Sturm-Liouville problems for λ associated to the solution $\varphi(x)$ and $\bar{\lambda}$ associated to the solution $\bar{\varphi}(x)$.

9. Let

$$u(x,t) = \frac{1}{2}\left(g(x+t) - g(t-x) + \int_{t-x}^{x+t} h(y)\ dy\right)$$

for some twice differentiable function g and continuous function h. Show that $u_{tt} = u_{xx}$
HINT: Just compute the derivatives and see what happens.