Name:____________________________________

Math 4163 Section 001 | *Practice Final Exam* | *November 23, 2014*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit!

1. Solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} u_t = ku_{xx} \\ u(-L,t) = u(L,t) \text{ and } u_x(-L,t) = u_x(L,t) \\ u(x,0) = f(x) \end{cases}$$

You do not need to derive what the coefficients are in the resulting series solution, just state the formulas for them if you can.

2. Solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} u_t = ku_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

You do not need to derive what the coefficients are in the resulting series solution, just state the formulas for them if you can.

3. Given the following Sturm-Liouville problem

$$\begin{cases} L(h) + \lambda\sigma(x)h = 0 \\ h(a) = h(b) = 0 \end{cases}$$

with

$$L(h) = \frac{d}{dx}\left(p(x)\frac{dh}{dx}\right) + q(x)h$$

Show that the eigenfunctions are orthogonal, i.e. show that for $m \neq n$,

$$\int_a^b \varphi_n(x)\varphi_m(x)\sigma(x)dx = 0$$

HINT: You will need Lagrange's identity, $\int_a^b (uL(v) - vL(u))dx = p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b$ and consider for $m \neq n$ the two Sturm-Liouville problems for λ_n associated to the solution $\varphi_n(x)$ and λ_m associated to the solution $\varphi_m(x)$.

4. Given the following Sturm-Liouville problem

$$\begin{cases} L(h) + \lambda\sigma(x)h = 0 \\ h(a) = h(b) = 0 \end{cases}$$

with

$$L(h) = \frac{d}{dx}\left(p(x)\frac{dh}{dx}\right) + q(x)h$$

Show that the eigenvalues are real, i.e. show that for $\lambda = \bar{\lambda}$.

HINT: You will need Lagrange's identity, $\int_a^b (uL(v) - vL(u))dx = p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b$ and consider the two Sturm-Liouville problems for λ associated to the solution $\varphi(x)$ and $\bar{\lambda}$ associated to the solution $\bar{\varphi}(x)$.

5. Using the eigenfunction expansion method, solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} u_t = ku_{xx} + Q(x,t) \\ u(0,t) = A(t), \;\; u(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$$

HINT: You will need Lagrange's identity, $\displaystyle\int_0^L \left(c\frac{d^2h}{dx^2} - h\frac{d^2c}{dx^2} \right) \, dx = \left(c\frac{dh}{dx} - h\frac{dc}{dx} \right)\Big|_0^L$.

6. Using the eigenfunction expansion method, solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} u_t = ku_{xx} + Q(x,t) \\ u_x(0,t) = A(t), \;\; u_x(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$$

HINT: You will need Lagrange's identity, $\displaystyle\int_0^L \left(c\frac{d^2h}{dx^2} - h\frac{d^2c}{dx^2} \right) \, dx = \left(c\frac{dh}{dx} - h\frac{dc}{dx} \right)\Big|_0^L$.

7. Using the Fourier transform, solve the following PDE subject to the following initial condition on the unbounded domain:

$$\begin{cases} u_t = ku_{xx} + cu_x \\ u(x,0) = f(x) \ , \ \ x \in \mathbb{R} \end{cases}$$

8. Using the Fourier transform, solve the following PDE subject to the following initial condition on the unbounded domain:

$$\begin{cases} u_t = ku_{xx} - cu \\ u(x,0) = f(x) \ , \ \ x \in \mathbb{R} \end{cases}$$

9. Using the Fourier transform, solve the following PDE subject to the following initial condition on the unbounded domain:

$$\begin{cases} u_t = ku_{xx} + q(x,t) \\ u(x,0) = f(x) \ , \ \ x \in \mathbb{R} \end{cases}$$

HINT: After you solved the first order ODE, use the fact that $e^a e^b = e^{a+b}$ and when computing the inverse Fourier transform use the fact that

$$\mathcal{F}^{-1}\left[\int F(\omega, s)\ ds\right] = \int \mathcal{F}^{-1}\left[F(\omega, s)\right]\ ds$$

10. Using the Fourier transform, solve the following PDE subject to the following initial condition on the unbounded domain:

$$\begin{cases} u_t = ku_{xxx} \\ u(x,0) = f(x) \ , \ \ x \in \mathbb{R} \end{cases}$$

HINT: You will need the definition of the Airy function, $Ai(x)$:

$$Ai(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\frac{\omega^3}{3}} e^{-i\omega x}\ d\omega = \frac{1}{\pi}\int_0^{\infty} \cos\left(\frac{\omega^3}{3} + x\omega\right)\ d\omega$$

11. Consider the following Schrödinger equation:

$$\begin{cases} \dfrac{d^2y}{dx^2} + (\lambda - V(x))\, y = 0 \\ y(0) = 0 \ , \ \ y \to 0 \text{ as } x \to \pm\infty \end{cases}$$

Show that the eigenfunctions from the discrete spectrum are orthogonal to the eigenfunctions from the continuous spectrum. That is if $-\kappa_n^2$ is from the discrete spectrum with eigenfunctions $\varphi_n(x)$ and if μ^2 is from the continuous spectrum with eigenfunctions $\nu(x)$ then

$$\int_{\mathbb{R}} \varphi_n(x)\nu(x)\ dx = 0$$

HINT: You will need Lagrange's identity, $\displaystyle\int_{\mathbb{R}} \left(c\frac{d^2h}{dx^2} - h\frac{d^2c}{dx^2} \right)\ dx = \left(c\frac{dh}{dx} - h\frac{dc}{dx} \right)\Big|_{-\infty}^{\infty}$.

12. Consider the following Schrödinger equation:

$$\begin{cases} \dfrac{d^2y}{dx^2} + (\lambda - V(x))\, y = 0 \\ y(0) = 0 \ , \ \ y \to 0 \text{ as } x \to \pm\infty \end{cases}$$

Show that the eigenfunctions from the discrete spectrum are orthogonal. That is if $-\kappa_n^2$ has the eigenfunction $\varphi_n(x)$ and if $-\kappa_m^2$ has the eigenfunction $\varphi_m(x)$ then

$$\int_{\mathbb{R}} \varphi_n(x)\varphi_m(x)\ dx = 0$$

HINT: You will need Lagrange's identity, $\displaystyle\int_{\mathbb{R}} \left(c\frac{d^2h}{dx^2} - h\frac{d^2c}{dx^2} \right)\ dx = \left(c\frac{dh}{dx} - h\frac{dc}{dx} \right)\Big|_{-\infty}^{\infty}$.

13. Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$, that is $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and denote the norm(lentgh) of $\mathbf{x}$ by $\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. For t a real number and $\mathbf{y}$ also a vector in $\mathbb{R}^n$, define:

$$u(\mathbf{x}, t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} g(\mathbf{y}) e^{-\frac{\|\mathbf{x}-\mathbf{y}\|^2}{4t}} \, d\mathbf{y}$$

Show that $u(\mathbf{x}, t)$ solves the heat equation $u_t = \Delta u$ where

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}$$