MATH 4163 Pr. Ex 3 sol'ns

1) $\begin{cases} \Delta v + \lambda v = 0 & \text{in } \Omega \\ v|_{\partial\Omega} = 0 \end{cases}$

let $\lambda_1 \neq \lambda_2$ be eigenvalues then $\lambda_1 \to \varphi_1$ and $\lambda_2 \to \varphi_2$ consider

$$\begin{cases} \Delta\varphi_1 = -\lambda_1\varphi_1 \\ \Delta\varphi_2 = -\lambda_2\varphi_2 \end{cases} \Rightarrow \begin{cases} \varphi_2\Delta\varphi_1 = -\lambda_1\varphi_1\varphi_2 \\ \varphi_1\Delta\varphi_2 = -\lambda_2\varphi_1\varphi_2 \end{cases} \Rightarrow \varphi_2\Delta\varphi_1 - \varphi_1\Delta\varphi_2 = (\lambda_2-\lambda_1)\varphi_1\varphi_2$$

then $(\lambda_2-\lambda_1)\iint_\Omega \varphi_1\varphi_2\,dA = \iint_\Omega(\varphi_2\Delta\varphi_1 - \varphi_1\Delta\varphi_2)\,dA \underset{\substack{\uparrow \\ \text{Lagrange Id.}}}{=} \int_{\partial\Omega}(\varphi_2\nabla\varphi_1 - \varphi_1\nabla\varphi_2)\cdot\vec{n}\,ds := A$

but $\varphi_1|_{\partial\Omega} = \varphi_2|_{\partial\Omega} = 0$ so $A \equiv 0$ $\Rightarrow (\lambda_2-\lambda_1)\iint_\Omega \varphi_1\varphi_2\,dA = 0$ but $\lambda_1 \neq \lambda_2$

so $\iint_\Omega \varphi_1\varphi_2\,dA = 0$

2.) $\begin{cases} \Delta v + \lambda v = 0 & \text{in } \Omega \\ (\nabla v\cdot\vec{n})|_{\partial\Omega} = 0 \end{cases}$

let $\lambda_1 \neq \lambda_2$ be eigenvalues then $\lambda_1 \to \varphi_1$ and $\lambda_2 \to \varphi_2$ consider

$$\begin{cases} \Delta\varphi_1 = -\lambda_1\varphi_1 \\ \Delta\varphi_2 = -\lambda_2\varphi_2 \end{cases} \Rightarrow \begin{cases} \varphi_2\Delta\varphi_1 = -\lambda_1\varphi_1\varphi_2 \\ \varphi_1\Delta\varphi_2 = -\lambda_2\varphi_1\varphi_2 \end{cases} \Rightarrow \varphi_2\Delta\varphi_1 - \varphi_1\Delta\varphi_2 = (\lambda_2-\lambda_1)\varphi_1\varphi_2$$

so $\iint_\Omega (\lambda_2-\lambda_1)\varphi_1\varphi_2\,dA = \iint_\Omega(\varphi_2\Delta\varphi_1 - \varphi_1\Delta\varphi_2)\,dA \underset{\substack{\uparrow \\ \text{Lagrange Id}}}{=} \int_{\partial\Omega}(\varphi_2\nabla\varphi_1 - \varphi_1\nabla\varphi_2)\cdot\vec{n}\,ds := A$

but $\nabla\varphi_1\cdot\vec{n}|_{\partial\Omega} = \nabla\varphi_2\cdot\vec{n}|_{\partial\Omega} = 0$ so $A \equiv 0$ $\Rightarrow (\lambda_2-\lambda_1)\iint_\Omega \varphi_1\varphi_2\,dA = 0$ but $\lambda_2 \neq \lambda_1$

so $\iint_\Omega \varphi_1\varphi_2\,dA = 0$

3.) $(x^3+x^2)\frac{d^2y}{dx^2} + \left(\frac{3}{16}+x^4\right)y = 0$

near zero what Euler type eq so $x^3\frac{d^2y}{dx^2} \approx 0$ and $x^4 y \approx 0$

$\Rightarrow$ $x^2\frac{d^2y}{dx^2} + \frac{3}{16}y \approx 0$ let $y = x^p$ so $y' = px^{p-1}$ $y'' = p(p-1)x^{p-2}$

$\Rightarrow$ $p(p-1)x^p + \frac{3}{16}x^p = 0$ $\Rightarrow$ $p(p-1) + \frac{3}{16} = 0$ $p^2 - p + \frac{3}{16} = 0$

$$p = \frac{1 \pm \sqrt{1-\frac{3}{4}}}{2} = \frac{1\pm\sqrt{\frac{1}{4}}}{2} = \frac{1\pm\frac{1}{2}}{2} = \frac{3}{4}, \frac{1}{4}$$

so $y \approx c_1 x^{\frac{3}{4}} + c_2 x^{\frac{1}{4}} + \ldots$ ← lower order terms

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4.) $(x^4+x^2)\frac{d^2y}{dx^2} + (x^8-3x)\frac{dy}{dx} + (7+x^5)y = 0$

near zero what in Euler type eq so $x^4\frac{d^2y}{dx^2}$, $x^8\frac{dy}{dx}$, $x^5 y \approx 0$

so $x^2y'' - 3xy' + 7y \approx 0$ let $y = x^p$ so $y' = px^{p-1}$, $y'' = p(p-1)x^{p-2}$

$\Rightarrow$ $p(p-1) - 3p + 7 = 0$ $\Rightarrow$ $p^2 - p - 3p + 7 = 0$ $p^2 - 4p + 7 = 0$

$$p = \frac{4\pm\sqrt{16-28}}{2} = \frac{4\pm\sqrt{-12}}{2} = 2 \pm i\sqrt{3}$$

so $y \approx x^2\left(c_1 x^{i\sqrt{3}} + c_2 x^{-i\sqrt{3}} + \ldots\right) = x^2\left(c_1\cos(\sqrt{3}\ln x) + c_2\sin(\sqrt{3}\ln x) + \ldots\right)$ ← lower order terms

5.) $u_t = \kappa u_{xx} + Q(x,t)$

$$\begin{cases} u(0,t) = A(t), \quad u(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$$

the associated homog. prob. $\begin{cases} \frac{d^2 h}{dx^2} + \lambda h = 0 \\ h(0) = h(L) = 0 \end{cases} \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2$, $\varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

$\frac{d\varphi_n}{dx} = \frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)$

Suppose $u(x,t) = \sum_{n=1}^{\infty} a_n(t)\varphi_n(x)$ and $Q(x,t) = \sum_{n=1}^{\infty} q_n(t)\varphi_n(x)$

then $a_n(t) = \frac{\int_0^L u\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx}$ and $q_n(t) = \frac{\int_0^L Q\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx}$

then $u_t = \sum_{n=1}^{\infty} \frac{da_n}{dt}\varphi_n$ and $u_t = \kappa u_{xx} + Q \Rightarrow \sum_{n=1}^{\infty} \frac{da_n}{dt}\varphi_n = \kappa u_{xx} + Q$

$$\Rightarrow \frac{da_n}{dt} = \frac{\kappa\int_0^L u_{xx}\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx} + \frac{\int_0^L Q\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx} = \kappa\frac{\int_0^L u_{xx}\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx} + q_n(t)$$

Study: $\int_0^L u_{xx}\varphi_n\,dx \overset{\text{Lagrange Id}}{=} \int_0^L u\frac{d^2\varphi_n}{dx^2}\,dx + \left(u_x\varphi_n - u\frac{d\varphi_n}{dx}\right)\Big|_0^L$ $= B(t)(-1)^n\frac{n\pi}{L} + A(t)\frac{n\pi}{L}$

$$= -\lambda_n\int_0^L u\varphi_n\,dx + \frac{n\pi}{L}\left(B(t)(-1)^n + A(t)\right)$$

So $\frac{\int_0^L u_{xx}\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx} = -\lambda_n\frac{\int_0^L u\varphi_n\,dx}{\int_0^L \varphi_n^2\,dx} + \frac{2n\pi}{L^2}\left(B(t)(-1)^n + A(t)\right) = -\lambda_n a_n(t) + D(t)$

So $\frac{da_n}{dt} = -\lambda_n\kappa a_n(t) + \kappa D(t) + q_n(t) = -\lambda_n\kappa a_n(t) + F(t)$

$$\Rightarrow a_n(t) = a_n(0)e^{-\lambda_n\kappa t} + e^{-\lambda_n\kappa t}\int_0^t F(\tau)e^{\lambda_n\kappa\tau}\,d\tau \quad \text{w/ } a_n(0) = \frac{\int_0^L f(x)\varphi_n(x)\,dx}{\int_0^L \varphi_n^2(x)\,dx}$$

6.) $\begin{cases} u_t = \kappa u_{xx} + Q(x,t) \\ u_x(0,t) = A(t), \quad u_x(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$

assoc. homogen eq.

$$\begin{cases} \frac{d^2h}{dx^2} + \lambda h = 0 \\ \frac{dh}{dx}(0) = \frac{dh}{dx}(L) = 0 \end{cases} \Rightarrow \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad \varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right), \quad \frac{d\varphi_n}{dx} = -\frac{n\pi}{L}\sin\left(\frac{n\pi x}{L}\right)$$

s'pose $u(x,t) = \sum_{n=0}^{\infty} a_n(t)\varphi_n(x)$ and $Q(x,t) = \sum_{n=0}^{\infty} q_n(t)\varphi_n(x)$

then $a_n(t) = \dfrac{\int_0^L u\varphi_n\,dx}{\int_0^L \varphi_n^2 dx}$, $q_n(t) = \dfrac{\int_0^L Q\varphi_n\,dx}{\int_0^L \varphi_n^2 dx}$

then $u_t = \sum_{n=0}^{\infty} \frac{da_n}{dt}\varphi_n(x)$ OTOH $u_t = \kappa u_{xx} + Q \Rightarrow \sum_{n=0}^{\infty} \frac{da_n}{dt}\varphi_n = \kappa u_{xx} + Q$

$$\Rightarrow \frac{da_n}{dt} = \frac{\kappa\int_0^L u_{xx}\varphi_n\,dx}{\int_0^L \varphi_n^2 dx} + \frac{\int_0^L Q\varphi_n\,dx}{\int_0^L \varphi_n^2 dx} = \kappa\frac{\int_0^L u_{xx}\varphi_n\,dx}{\int_0^L \varphi_n^2 dx} + q_n(t)$$

Study: $\int_0^L u_{xx}\varphi_n\,dx \underset{\text{Lagrange Id}}{=} \int_0^L u\frac{d^2\varphi_n}{dx^2}dx + \left(u_x\varphi_n - u\frac{d\varphi_n}{dx}\right)\Big|_0^L = B(t)(-1)^n - A(t) + \int_0^L u\frac{d^2\varphi_n}{dx^2}dx$

$= -\lambda_n\int_0^L u\varphi_n\,dx + B(t)(-1)^n - A(t)$ so $\dfrac{\int_0^L u_{xx}\varphi_n dx}{\int_0^L \varphi_n^2 dx} = \dfrac{-\lambda_n\int_0^L u\varphi_n dx}{\int_0^L \varphi_n^2 dx} + \frac{2}{L}\left(B(t)(-1)^n - A(t)\right)$

$= -\lambda_n a_n(t) + D(t) \Rightarrow \frac{da_n}{dt} = -\lambda_n\kappa a_n + \kappa D(t) + q_n(t) = -\lambda_n\kappa a_n + F(t)$

So $a_n(t) = a_n(0)e^{-\lambda_n\kappa t} + e^{-\lambda_n\kappa t}\int_0^t F(\tau)e^{\lambda_n\kappa\tau}d\tau$ w/ $a_n(0) = \dfrac{\int_0^L f(x)\varphi_n(x)dx}{\int_0^L \varphi_n^2(x)dx}$

2.) recall $F(\omega) = \widehat{f(x)} = \frac{1}{2\pi}\int_{\mathbb{R}} f(x)\, e^{i\omega x}\, dx$

so $$\widehat{\left(\frac{\partial^3 f}{\partial x^3}\right)} = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\partial^3 f}{\partial x^3} e^{i\omega x}\, dx = \frac{1}{2\pi}\left(\frac{\partial^2 f}{\partial x^2} e^{i\omega x}\Big|_{-\infty}^{\infty} - \frac{\partial f}{\partial x}\, i\omega e^{i\omega x}\Big|_{-\infty}^{\infty} + f(-i\omega)^2 e^{i\omega x}\Big|_{-\infty}^{\infty} + \int_{\mathbb{R}} (-i\omega)^3 f(x)\, e^{i\omega x}\, dx \right)$$

via by parts 3 times

since $f \to 0$ and $\frac{\partial^n f}{\partial x^n} \to 0$ for all $n \geq 1$ as $x \to \pm\infty$ all the bdry terms are zero so

$$\widehat{\left(\frac{\partial^3 f}{\partial x^3}\right)} = \frac{1}{2\pi}\int_{\mathbb{R}} (-i\omega)^3 f(x) e^{i\omega x}\, dx = (-i\omega)^3 \hat{f}$$

8.) compute

$$\widehat{\left(\frac{\partial^7 f}{\partial x^7}\right)} = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{\partial^7 f}{\partial x^7} e^{i\omega x}\, dx = \frac{1}{2\pi}\left(\frac{\partial^6 f}{\partial x^6} e^{i\omega x}\Big|_{-\infty}^{\infty} - \cdots + (-i\omega)^6 f e^{i\omega x}\Big|_{-\infty}^{\infty} + \int_{\mathbb{R}} (-i\omega)^7 f(x)\, e^{i\omega x}\, dx \right)$$

via by parts 7 times

since $f \to 0$ and $\frac{\partial^n f}{\partial x^n} \to 0$ for all $n \geq 1$ as $x \to \pm\infty$ all the bdry terms are zero so

$$\widehat{\left(\frac{\partial^7 f}{\partial x^7}\right)} = \frac{1}{2\pi}\int_{\mathbb{R}} (-i\omega)^7 f(x) e^{i\omega x}\, dx = (-i\omega)^7 \hat{f}$$

9.) $u(x_1,\dots,x_n,t) = g(x_1 - b_1 t, \dots, x_n - b_n t) + \int_0^t f(x_1+(s-t)b_1, \dots, x_n+(s-t)b_n, s)\,ds$

let $v_j = x_j - b_j t$ for $j=1,\dots,n$ $\quad w_j = x_j + (s-t)b_j$ for $j=1,\dots,n$

then $\frac{\partial v_j}{\partial t} = -b_j$, $\frac{\partial v_j}{\partial x_j} = 1$ $\quad \frac{\partial w_j}{\partial t} = -b_j$, $\frac{\partial w_j}{\partial s} = b_j$, $\frac{\partial w_j}{\partial x_j} = 1$

so $u = g(v_1, v_2, \dots, v_n) + \int_0^t f(w_1,\dots,w_n,s)\,ds$

so $u_t = \frac{\partial g}{\partial v_1}\frac{\partial v_1}{\partial t} + \dots + \frac{\partial g}{\partial v_n}\frac{\partial v_n}{\partial t} + f(x_1,\dots,x_n,t) + \int_0^t \left(\frac{\partial f}{\partial w_1}\frac{\partial w_1}{\partial t} + \dots + \frac{\partial f}{\partial w_n}\frac{\partial w_n}{\partial t}\right) ds$

$$= \sum_{j=1}^n \frac{\partial g}{\partial v_j}(-b_j) + f + \int_0^t \sum_{j=1}^n \frac{\partial f}{\partial w_j}(-b_j)\,ds$$

recall $\nabla u = (u_{x_1}, u_{x_2}, \dots, u_{x_n}) = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n}\right)$

then $\frac{\partial u}{\partial x_j} = \frac{\partial g}{\partial v_1}\frac{\partial v_1}{\partial x_j} + \dots + \frac{\partial g}{\partial v_n}\frac{\partial v_1}{\partial x_j} + \int_0^t \left(\frac{\partial f}{\partial w_1}\frac{\partial w_1}{\partial x_j} + \dots + \frac{\partial f}{\partial w_n}\frac{\partial w_n}{\partial x_j}\right) ds$

$= \frac{\partial g}{\partial v_j}\frac{\partial v_j}{\partial x_j} + \int_0^t \frac{\partial f}{\partial w_j}\frac{\partial w_j}{\partial x_j}\,ds = \frac{\partial g}{\partial v_j} + \int_0^t \frac{\partial f}{\partial w_j}\,ds$ all others are zero

so $\nabla u \cdot \vec{b} = \sum_{j=1}^n \frac{\partial u}{\partial x_j} b_j = \sum_{j=1}^n \left[\frac{\partial g}{\partial v_j} b_j + \int_0^t \frac{\partial f}{\partial w_j} b_j\,ds\right]$

then $u_t + \nabla u \cdot \vec{b} = f + \underbrace{\sum_{j=1}^n \left[\frac{\partial g}{\partial v_j}(-b_j) + \int_0^t \frac{\partial f}{\partial w_j}(-b_j)\,ds\right] + \sum_{j=1}^n \left[\frac{\partial g}{\partial v_j} b_j + \int_0^t \frac{\partial f}{\partial w_j} b_j\,ds\right]}_{=0}$

$= f$