Name:______________________________

Math 4163 Section 001 *Practice Exam 3* *October 28, 2014*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit!

1. Given the following Sturm-Liouville problem

$$\begin{cases} \Delta v + \lambda v = 0 \\ v|_{\partial\Omega} = 0 \end{cases}$$

Show that the eigenfunctions are orthogonal, i.e. show that for two different eigenvalues $\lambda_1 \neq \lambda_2$,

$$\int\int_{\Omega} \varphi_1(x,y)\varphi_2(x,y)\, dA = 0$$

HINT: You will need Lagrange's identity, $\int\int_{\Omega}(u\Delta v - v\Delta u)\, dA = \int_{\partial\Omega}(u\nabla v - v\nabla u)\cdot \mathbf{n}\, ds$ and consider for the two Sturm-Liouville problems for λ_1 associated to the solution $\varphi_1(x,y)$ and λ_2 associated to the solution $\varphi_2(x,y)$.

2. Given the following Sturm-Liouville problem

$$\begin{cases} \Delta v + \lambda v = 0 \\ \nabla v \cdot \mathbf{n}|_{\partial\Omega} = 0 \end{cases}$$

Show that the eigenfunctions are orthogonal, i.e. show that for two different eigenvalues $\lambda_1 \neq \lambda_2$,

$$\int\int_{\Omega} \varphi_1(x,y)\varphi_2(x,y)\, dA = 0$$

HINT: You will need Lagrange's identity, $\int\int_{\Omega}(u\Delta v - v\Delta u)\, dA = \int_{\partial\Omega}(u\nabla v - v\nabla u)\cdot \mathbf{n}\, ds$ and consider for the two Sturm-Liouville problems for λ_1 associated to the solution $\varphi_1(x,y)$ and λ_2 associated to the solution $\varphi_2(x,y)$.

3. Given the following ODE:

$$(x^3 + x^2)\frac{d^2y}{dx^2} + \left(\frac{3}{16} + x^4\right) y = 0$$

What is the expected approximate behavior of the solutions near $x = 0$.

4. Given the following ODE:

$$(x^4 + x^2)\frac{d^2y}{dx^2} + (x^8 - 3x)\frac{dy}{dx} + (7 + x^5)y = 0$$

What is the expected approximate behavior of the solutions near $x = 0$.

5. Using the eigenfunction expansion method, solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} u_t = ku_{xx} + Q(x,t) \\ u(0,t) = A(t),\ u(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$$

HINT: You will need Lagrange's identity, $\displaystyle\int_0^L \left(c\frac{d^2h}{dx^2} - h\frac{d^2c}{dx^2}\right)\, dx = \left(c\frac{dh}{dx} - h\frac{dc}{dx}\right)\Big|_0^L$.

6. Using the eigenfunction expansion method, solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} u_t = ku_{xx} + Q(x,t) \\ u_x(0,t) = A(t), \ u_x(L,t) = B(t) \\ u(x,0) = f(x) \end{cases}$$

HINT: You will need Lagrange's identity, $\displaystyle\int_0^L \left(c\frac{d^2h}{dx^2} - h\frac{d^2c}{dx^2} \right) \, dx = \left(c\frac{dh}{dx} - h\frac{dc}{dx} \right)\Big|_0^L$.

7. Using the definition of the Fourier transform, show that

$$\widehat{\left(\frac{\partial^3 f}{\partial x^3}\right)} = (-i\omega)^3 \widehat{f}$$

You may assume that f and all its derivatives in x vanish as $x \to \pm\infty$.

8. Using the definition of the Fourier transform, show that

$$\widehat{\left(\frac{\partial^7 f}{\partial x^7}\right)} = (-i\omega)^7 \widehat{f}$$

You may assume that f and all its derivatives in x vanish as $x \to \pm\infty$.

9. Let $\mathbf{b}$ be a vector in $\mathbb{R}^n$, that is $\mathbf{b} = (b_1, \ldots, b_n)$. Also let g be a differentiable function in $\mathbb{R}^n$. Define

$$u(x_1, \ldots, x_n, t) = g(x_1 - b_1 t, \ldots, x_n - b_n t) + \int_0^t f(x_1 + (s-t)b_1, \ldots, x_n + (s-t)b_n, s)\, ds$$

Show that $u_t + \mathbf{b} \cdot \nabla u = f$ HINT: Just compute the derivatives and see what happens.