Practice Exam 2 Soln's

1) $\begin{cases} au_{tt} = bu_{xx} - cu_t \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \end{cases}$ $\qquad c^2 < \frac{4\pi^2 ab}{L^2}$

s'pose $u(x,t) = g(t)h(x)$ so $\Rightarrow$ $ah\frac{d^2g}{dt^2} = bg\frac{d^2h}{dx^2} - ch\frac{dg}{dt}$

$\Rightarrow h\left(a\frac{d^2g}{dt^2} + c\frac{dg}{dt}\right) = bg\frac{d^2h}{dx^2}$ $\Rightarrow$ $\frac{1}{bg}\left(a\frac{d^2g}{dt^2} + c\frac{dg}{dt}\right) = \frac{1}{h}\frac{d^2h}{dx^2} = -\lambda$

(1) $a\frac{d^2g}{dt^2} + c\frac{dg}{dt} + \lambda bg = 0$ $\qquad$ (2) $\begin{cases} \frac{d^2h}{dx^2} + \lambda h = 0 \\ h(0) = h(L) = 0 \end{cases}$

for (2) $r = \pm\sqrt{-\lambda}$ $\quad$ if $\lambda = 0$: $\Rightarrow$ $h = c_1 + c_2 x$ $\quad 0 = h(0) = c_1$ and $0 = h(L) = c_2 L$

$\Rightarrow c_1 = c_2 = 0$

if $\lambda < 0$: then set $-\lambda = s^2$ $\Rightarrow$ $r = \pm s$ and $h(x) = c_1 e^{sx} + c_2 e^{-sx}$ $\quad 0 = h(0) = c_1 + c_2$

so $c_2 = -c_1$ $\Rightarrow$ $h(x) = c_1(e^{sx} - e^{-sx})$ $\quad 0 = h(L) = c_1(e^{sL} - e^{-sL})$ but $e^{sL} \neq e^{-sL}$

$\Rightarrow c_1 = 0 \Rightarrow c_2 = 0$

if $\lambda > 0$: $\quad h(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$ $\quad 0 = h(0) = c_1$ and $0 = h(L) = c_2\sin(\sqrt{\lambda}L)$

$\Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2$ and $h_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ $\quad n \geq 1$

then for (1) $\quad ar^2 + cr + \left(\frac{n\pi}{L}\right)^2 b = 0$ $\quad$ so $r = \dfrac{-c \pm \sqrt{c^2 - 4ab\left(\frac{n\pi}{L}\right)^2}}{2a}$

since $c^2 < \frac{4\pi^2 ab}{L^2} < \frac{4\pi^2 n^2 ab}{L^2}$ $\Rightarrow$ $c^2 - 4ab\left(\frac{n\pi}{L}\right)^2 < 0$ $\quad$ set $-\varepsilon^2 = c^2 - 4ab\left(\frac{n\pi}{L}\right)^2$
$n \geq 1$

$\Rightarrow r = \frac{-c}{2a} \pm \frac{i\varepsilon}{2a} := -\gamma \pm i\delta$ $\Rightarrow$ $g(t) = e^{-\gamma t}(c_1\cos(\delta t) + c_2\sin(\delta t))$

so $\quad u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\gamma t}\cos(\delta t)\sin\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n e^{-\gamma t}\sin(\delta t)\sin\left(\frac{n\pi x}{L}\right)$

$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ $\quad$, $\quad -\gamma a_n + \delta b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$n \geq 1$ $\qquad\qquad n \geq 1$

2) $\begin{cases} a u_{tt} = b u_{xx} - c u_t \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \end{cases}$ $\qquad c^2 < \dfrac{4\pi^2 a b}{L^2}$

s'pose $u(x,t) = h(x)g(t) \Rightarrow \dfrac{1}{bg}\left(a\dfrac{d^2 g}{dt^2} + c\dfrac{dg}{dt}\right) = \dfrac{1}{h}\dfrac{d^2h}{dx^2} = -\lambda$

① $a\dfrac{d^2g}{dt^2} + c\dfrac{dg}{dt} + \lambda b g = 0$ $\qquad$ ② $\begin{cases} \dfrac{d^2h}{dx^2} + \lambda h = 0 \\ \dfrac{dh}{dx}(0) = \dfrac{dh}{dx}(L) = 0 \end{cases}$

for ② $r = \pm\sqrt{-\lambda}$ if $\underline{\lambda = 0}$: $\Rightarrow h(x) = c_1 + c_2 x$ $\quad \dfrac{dh}{dx} = c_2$ $\quad \dfrac{dh}{dx}(0) = \dfrac{dh}{dx}(L) = 0 \Rightarrow c_2 = 0$

so $h = c_1$

if $\underline{\lambda < 0}$: then set $-\lambda = s^2 \Rightarrow r = \pm s$ so $h(x) = c_1 e^{sx} + c_2 e^{-sx}$, $\dfrac{dh}{dx} = s(c_1 e^{sx} - c_2 e^{-sx})$

$0 = \dfrac{dh}{dx}(0) = c_1 - c_2 \Rightarrow c_1 = c_2 \Rightarrow h(x) = c_1 s(e^{sx} - e^{-sx})$ $\quad 0 = \dfrac{dh}{dx}(L) = c_1 s(e^{sL} - e^{-sL})$

but $s \neq 0$, $e^{sL} \neq e^{-sL} \Rightarrow c_1 = 0 \Rightarrow c_2 = 0$

if $\underline{\lambda > 0}$: $h(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$ $\quad \dfrac{dh}{dx} = \sqrt{\lambda}(-c_1\sin(\sqrt{\lambda}x) + c_2\cos(\sqrt{\lambda}x))$

$0 = \dfrac{dh}{dx}(0) = \sqrt{\lambda}c_2 \Rightarrow c_2 = 0$ so $0 = \dfrac{dh}{dx}(L) = -\sqrt{\lambda}c_1\sin(\sqrt{\lambda}L) \Rightarrow \lambda_n = \left(\dfrac{n\pi}{L}\right)^2$

and $h_n(x) = \cos\left(\dfrac{n\pi x}{L}\right)$ $\quad n \geq 1$

for ① $ar^2 + cr + \left(\dfrac{n\pi}{L}\right)^2 b = 0$ so $r = \dfrac{-c \pm \sqrt{c^2 - 4ab\left(\frac{n\pi}{L}\right)^2}}{2a}$

since $c^2 < \dfrac{4\pi^2 ab}{L^2} < \dfrac{4\pi^2 ab n^2}{L^2} \Rightarrow c^2 - 4ab\left(\dfrac{n\pi}{L}\right)^2 < 0$ set $-\varepsilon^2 = c^2 - 4ab\left(\dfrac{n\pi}{L}\right)^2$

$\Rightarrow r = -\dfrac{c}{2a} \pm \dfrac{i\varepsilon}{2a} := -\gamma \pm i\delta \Rightarrow g(t) = e^{-\gamma t}(c_1\cos(\delta t) + c_2\sin(\delta t))$

so $u(x,t) = \displaystyle\sum_{n=0}^{\infty} a_n e^{-\gamma t}\cos(\delta t)\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=0}^{\infty} b_n e^{-\gamma t}\sin(\delta t)\cos\left(\frac{n\pi x}{L}\right)$

$a_n = \dfrac{2}{L}\displaystyle\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ $\quad n \geq 1$, $\quad a_0 = \dfrac{1}{L}\displaystyle\int_0^L f(x)\,dx$

$-\gamma a_n + \delta b_n = \dfrac{2}{L}\displaystyle\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)dx$ $\quad n \geq 1$, $\quad -\gamma a_0 + \delta b_0 = \dfrac{1}{L}\displaystyle\int_0^L g(x)\,dx$

3.) $\begin{cases} ab\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(c\dfrac{\partial u}{\partial x}\right) + \alpha u \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$

s'pose $u(x,t) = h(x)g(t)$ $\Rightarrow$ $abh\dfrac{dg}{dt} = \dfrac{\partial}{\partial x}\left(cg\dfrac{dh}{dx}\right) + \alpha hg$

$\Rightarrow$ $\dfrac{1}{g}\dfrac{dg}{dt} = \dfrac{1}{ab}\dfrac{1}{h}\left[\dfrac{d}{dx}\left(c\dfrac{dh}{dx}\right) + \alpha h\right] = -\lambda$

$\Rightarrow$ ① $\dfrac{dg}{dt} = -\lambda g$ gives $g = e^{-\lambda t}$ ② $\begin{cases} \dfrac{d}{dx}\left(c\dfrac{dh}{dx}\right) + \alpha h + ab\lambda h = 0 \\ h(0) = h(L) = 0 \end{cases}$

② is a S-L prob w/ Dirichlet bndy condns and weight $\sigma = ab$

so there are eigenvalues and fncs λ_n, $\varphi_n(x)$ by S-L Thm

so $u(x,t) = \displaystyle\sum_{n=0}^{\infty} a_n e^{-\lambda_n t}\varphi_n(x)$ and $a_n = \dfrac{\int_0^L f(x)\varphi_n(x)\,ab\,dx}{\int_0^L \varphi_n^2(x)\,ab\,dx}$

$u(x,t) \longrightarrow a_0$ as $t \to \infty$.

4.) $\begin{cases} ab\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(c\dfrac{\partial u}{\partial x}\right) + \alpha u \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$

s'pose $u(x,t) = h(x)g(t)$ $\Rightarrow$ $\dfrac{1}{g}\dfrac{dg}{dt} = \dfrac{1}{ab}\dfrac{1}{h}\left[\dfrac{d}{dx}\left(c\dfrac{dh}{dx}\right) + \alpha h\right] = -\lambda$

$\Rightarrow$ ① $\dfrac{dg}{dt} = -\lambda g$ gives $g = e^{-\lambda t}$ ② $\begin{cases} \dfrac{d}{dx}\left(c\dfrac{dh}{dx}\right) + \alpha h + ab\lambda h = 0 \\ \dfrac{dh}{dx}(0) = \dfrac{dh}{dx}(L) = 0 \end{cases}$

② is a S-L prob w/ Neumann bndy conditions w/ $\sigma = ab$

so by S-L Thm there are values and fncs λ_n, $\varphi_n(x)$

so $u(x,t) = \displaystyle\sum_{n=0}^{\infty} a_n e^{-\lambda_n t}\varphi_n(x)$ and $a_n = \dfrac{\int_0^L f\varphi_n\,ab\,dx}{\int_0^L \varphi_n^2\,ab\,dx}$

$u(x,t) \longrightarrow a_0$ as $t \to \infty$

5.) $\begin{cases} L(h) + \lambda \sigma h = 0 \\ h(a) = h(b) = 0 \end{cases}$

let λ_n, λ_m be eigenvalues w/ $\lambda_n \neq \lambda_m$ then $\lambda_n \leftrightarrow \varphi_n(x)$ and $\lambda_m \leftrightarrow \varphi_m(x)$

Consider $\begin{cases} L(\varphi_n) + \lambda_n \sigma \varphi_n = 0 \\ L(\varphi_m) + \lambda_m \sigma \varphi_n = 0 \end{cases} \Rightarrow \begin{cases} \varphi_m L(\varphi_n) + \lambda_n \varphi_n \varphi_m \sigma = 0 \\ \varphi_n L(\varphi_m) + \lambda_m \varphi_n \varphi_m \sigma = 0 \end{cases}$

$\Rightarrow \quad \varphi_m L(\varphi_n) - \varphi_n L(\varphi_m) = (\lambda_m - \lambda_n) \varphi_n \varphi_m \sigma$

$\Rightarrow \quad (\lambda_n - \lambda_m) \int_a^b \varphi_n \varphi_m \sigma \, dx = \int_a^b \left(\varphi_m L(\varphi_n) - \varphi_n L(\varphi_m) \right) dx$

via Lagrange
$= \left(p \, \varphi_m \frac{d\varphi_n}{dx} - \varphi_n \frac{d\varphi_m}{dx} \right) \Big|_a^b = 0$ since $\varphi_m(a) = \varphi_n(a) = \varphi_m(b) = \varphi_n(b) = 0$

$\Rightarrow \quad (\lambda_n - \lambda_m) \int_a^b \varphi_n \varphi_m \sigma \, dx = 0$ but $\lambda_n \neq \lambda_m \Rightarrow \int_a^b \varphi_n \varphi_m \sigma \, dx = 0$

6.) $\begin{cases} L(h) + \lambda \sigma h = 0 \\ h(a) = h(b) = 0 \end{cases}$

let $\lambda \leftrightarrow \varphi(x)$ and $\bar{\lambda} \leftrightarrow \bar{\varphi}(x)$ then consider $\begin{cases} L(\varphi) + \lambda \sigma \varphi = 0 \\ L(\bar{\varphi}) + \bar{\lambda} \sigma \bar{\varphi} = 0 \end{cases}$

$\Rightarrow \begin{cases} \bar{\varphi} L(\varphi) = -\lambda \sigma \varphi \bar{\varphi} \\ \varphi L(\bar{\varphi}) = -\bar{\lambda} \sigma \varphi \bar{\varphi} \end{cases} \Rightarrow \bar{\varphi} L(\varphi) - \varphi L(\bar{\varphi}) = (\bar{\lambda} - \lambda) \varphi \bar{\varphi} \sigma$ recall $\varphi \bar{\varphi} = |\varphi|^2$

so $(\bar{\lambda} - \lambda) \int_a^b |\varphi|^2 \sigma \, dx = \int_a^b \left(\bar{\varphi} L(\varphi) - \varphi L(\bar{\varphi}) \right) dx \overset{\text{via Lagrange}}{=} \left(p \, \bar{\varphi} \frac{d\varphi}{dx} - \varphi \frac{d\bar{\varphi}}{dx} \right) \Big|_a^b = 0$

since $\varphi(a) = \varphi(b) = 0$ and since a, b real $\bar{\varphi}(a) = \bar{\varphi}(b) = 0$

so $(\bar{\lambda} - \lambda) \int_a^b |\varphi|^2 \sigma \, dx = 0$ but $\int_a^b |\varphi|^2 \sigma \, dx \geq 0$ and $= 0$ only if $\varphi = 0$

so $\Rightarrow \bar{\lambda} = \lambda$ hence real eigenvalues.

7.) $\begin{cases} u_t = \kappa \Delta u \\ u(0,y,t) = u(L,y,t) = 0 \\ u(x,0,t) = u(x,H,t) = 0 \\ u(x,y,0) = \alpha(x,y) \end{cases}$

s'pose $u(x,y,t) = f(t)\,v(x,y)$ $\longrightarrow$ ① $\frac{df}{dt} = -\lambda\kappa f$ $\Rightarrow$ $f(t) = e^{-\lambda\kappa t}$ ② $\begin{cases} \Delta v + \lambda v = 0 \\ v(0,y) = v(L,y) = 0 \\ v(x,0) = v(x,H) = 0 \end{cases}$

for ② s'pose $v(x,y) = h(x)g(y)$ $\Rightarrow$ $g\frac{d^2h}{dx^2} + h\frac{d^2g}{dy^2} = -\lambda hg$

$\Rightarrow \frac{1}{h}\frac{d^2h}{dx^2} + \frac{1}{g}\frac{d^2g}{dy^2} = -\lambda \Rightarrow \underbrace{\frac{1}{h}\frac{d^2h}{dx^2}}_{x} = \underbrace{-\frac{1}{g}\frac{d^2g}{dy^2} - \lambda}_{y} = -\mu$

(1*) $\begin{cases} \frac{d^2h}{dx^2} + \mu h = 0 \\ h(0) = h(L) = 0 \end{cases}$ (2*) $\begin{cases} \frac{d^2g}{dy^2} + (\lambda - \mu)g = 0 \\ g(0) = g(H) = 0 \end{cases}$

(1*) yields $\mu_n = \left(\frac{n\pi}{L}\right)^2$ & $h_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ $n \geq 1$ for (2*) set $\lambda_{mn} - \mu_n = \left(\frac{m\pi}{H}\right)^2$ $m \geq 1$ & $g_n(y) = \sin\left(\frac{m\pi y}{H}\right)$

so for ② $\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$ & $\varphi_{mn}(x,y) = \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$

so $u(x,y,t) = \sum_{m,n=1}^{\infty} A_{mn}\, e^{-\lambda_{mn}\kappa t} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)$

& $A_{mn} = \dfrac{\int_0^L\int_0^H \alpha(x,y)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi y}{H}\right)dy\,dx}{\int_0^L\int_0^H \sin^2\left(\frac{n\pi x}{L}\right)\sin^2\left(\frac{m\pi y}{H}\right)dy\,dx}$

8.) $\begin{cases} u_t = \kappa \Delta u \\ u(0,y,t) = u(L,y,t) = 0 \\ u_y(x,0,t) = u_y(x,H,t) = 0 \\ u(x,y,0) = \alpha(x,y) \end{cases}$

S'pose $u(x,y,t) = f(t)\, v(x,y) \rightarrow$ ① $\frac{df}{dt} = -\kappa\lambda f \Rightarrow f(t) = e^{-\kappa\lambda t}$

② $\begin{cases} \Delta v + \lambda v = 0 \\ v(0,y) = v(L,y) = 0 \\ v_y(x,0) = v_y(x,H) = 0 \end{cases}$

for ② s'pose $v(x,y) = h(x)g(y) \Rightarrow g\frac{d^2h}{dx^2} + h\frac{d^2g}{dy^2} = -\lambda h g$

$\Rightarrow \frac{1}{h}\frac{d^2h}{dx^2} + \frac{1}{g}\frac{d^2g}{dy^2} = -\lambda \Rightarrow \underbrace{\frac{1}{h}\frac{d^2h}{dx^2}}_{x} = \underbrace{-\frac{1}{g}\frac{d^2g}{dy^2} - \lambda}_{y} = -\mu$

(1*) $\begin{cases} \frac{d^2h}{dx^2} + \mu h = 0 \\ h(0) = h(L) = 0 \end{cases}$ (2*) $\begin{cases} \frac{d^2g}{dy^2} + (\lambda - \mu) g = 0 \\ \frac{dg}{dy}(0) = \frac{dg}{dy}(H) = 0 \end{cases}$

(1*) yields $\mu_n = \left(\frac{n\pi}{L}\right)^2$, $h_n(x) = \sin\left(\frac{n\pi x}{L}\right)$ $n \geq 1$

for (2*) get $\lambda_{mn} - \mu_n = \left(\frac{m\pi}{H}\right)^2$ $g_n(y) = \cos\left(\frac{m\pi y}{H}\right)$ $m \geq 0$

so $\lambda_{mn} = \left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{H}\right)^2$ $n \geq 1,\ m \geq 0$ $\varphi_{mn}(x,y) = \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)$

so $u(x,y,t) = \sum_{\substack{m=0 \\ n=1}}^{\infty} a_{mn}\, e^{-\lambda_{mn}\kappa t} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right)$

$$a_{mn} = \frac{\int_0^L \int_0^H \alpha(x,y) \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi y}{H}\right) dy\,dx}{\int_0^L \int_0^H \sin^2\left(\frac{n\pi x}{L}\right)\cos^2\left(\frac{m\pi y}{H}\right) dy\,dx} \quad m,n \geq 1$$

$$a_{0n} = \frac{\int_0^L \int_0^H \alpha(x,y) \sin\left(\frac{n\pi x}{L}\right) dy\,dx}{\int_0^L \int_0^H \sin^2\left(\frac{n\pi x}{L}\right) dy\,dx}$$

9.) $u(x,t) = \frac{1}{2}\left(g(x+ct) + g(x-ct) + \int_{x-ct}^{x+ct} h(y)\,dy \right)$

let $v = x+ct$ and $w = x-ct$ then $\frac{\partial v}{\partial x} = 1$ $\frac{\partial v}{\partial t} = c$

$\frac{\partial w}{\partial x} = 1$ $\frac{\partial w}{\partial t} = -c$

so $u(x,t) = \frac{1}{2}\left(g(v) + g(w) + \int_v^w h(y)\,dy \right)$

then $u_t = \frac{1}{2}\left(\frac{dg}{dv}\frac{\partial v}{\partial t} + \frac{dg}{dw}\frac{\partial w}{\partial t} + h(w)\frac{\partial w}{\partial t} - h(v)\frac{\partial v}{\partial t} \right)$

$= \frac{1}{2}\left(\frac{dg}{dv}c - \frac{dg}{dw}c - h(w)c - h(v)c \right)$

$u_{tt} = \frac{1}{2}\left(\frac{d^2g}{dv^2}\frac{\partial v}{\partial t}c - \frac{d^2g}{dw^2}\frac{\partial w}{\partial t}c - \frac{dh}{dw}\frac{\partial w}{\partial t}c - \frac{dh}{dv}\frac{\partial v}{\partial t}c \right)$

$= \frac{c^2}{2}\left(\frac{d^2g}{dv^2} + \frac{d^2g}{dw^2} + \frac{dh}{dw} - \frac{dh}{dv} \right)$

$u_x = \frac{1}{2}\left(\frac{dg}{dv}\frac{\partial v}{\partial x} + \frac{dg}{dw}\frac{\partial w}{\partial x} + h(w)\frac{\partial w}{\partial x} - h(v)\frac{\partial v}{\partial x} \right)$

$= \frac{1}{2}\left(\frac{dg}{dv} + \frac{dg}{dw} + h(w) - h(v) \right)$

$u_{xx} = \frac{1}{2}\left(\frac{d^2g}{dv^2}\frac{\partial v}{\partial x} + \frac{d^2g}{dw^2}\frac{\partial w}{\partial x} + \frac{dh}{dw}\frac{\partial w}{\partial x} - \frac{dh}{dv}\frac{\partial v}{\partial x} \right) = \frac{1}{2}\left(\frac{d^2g}{dv^2} + \frac{d^2g}{dw^2} + \frac{dh}{dw} - \frac{dh}{dv} \right)$

comparing see get $u_{tt} = c^2 u_{xx}$