Name:__________________________________

Math 4163 Section 001 — *Practice Exam 2* — *October 9, 2014*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit!

1. Solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} au_{tt} = bu_{xx} - cu_t \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \text{ and } u_t(x,0) = g(x) \end{cases}$$

where a, b, and c are constants and $c^2 < \dfrac{4\pi^2 ab}{L^2}$. You do not need to derive what the coefficients are in the resulting series solution, just state the formulas for them if you can. Be sure to do the full analysis for for the underlying boundary value problem.

2. Solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} au_{tt} = bu_{xx} - cu_t \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = f(x) \text{ and } u_t(x,0) = g(x) \end{cases}$$

where a, b, and c are constants and $c^2 < \dfrac{4\pi^2 ab}{L^2}$. You do not need to derive what the coefficients are in the resulting series solution, just state the formulas for them if you can. Be sure to do the full analysis for for the underlying boundary value problem.

3. Solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} a(x)b(x)\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(c(x)\dfrac{\partial u}{\partial x}\right) + \alpha(x)u \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

What happens to $u(x,t)$ as $t \to \infty$? You do not need to derive what the coefficients are in the resulting series solution, just state the formulas for them if you can. Also don't forget to state that you are using the Sturm-Liouville Theorem when solving the eigenvalue problem.

4. Solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} a(x)b(x)\dfrac{\partial u}{\partial t} = \dfrac{\partial}{\partial x}\left(c(x)\dfrac{\partial u}{\partial x}\right) + \alpha(x)u \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

What happens to $u(x,t)$ as $t \to \infty$? You do not need to derive what the coefficients are in the resulting series solution, just state the formulas for them if you can. Also don't forget to state that you are using the Sturm-Liouville Theorem when solving the eigenvalue problem.

5. Given the following Sturm-Liouville problem

$$\begin{cases} L(h) + \lambda\sigma(x)h = 0 \\ h(a) = h(b) = 0 \end{cases}$$

with

$$L(h) = \frac{d}{dx}\left(p(x)\frac{dh}{dx}\right) + q(x)h$$

Show that the eigenfunctions are orthogonal, i.e. show that for $m \neq n$,

$$\int_a^b \varphi_n(x)\varphi_m(x)\sigma(x)dx = 0$$

HINT: You will need Lagrange's identity, $\displaystyle\int_a^b (uL(v) - vL(u))dx = p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b$ and consider for $m \neq n$ the two Sturm-Liouville problems for λ_n associated to the solution $\varphi_n(x)$ and λ_m associated to the solution $\varphi_m(x)$.

6. Given the following Sturm-Liouville problem

$$\begin{cases} L(h) + \lambda\sigma(x)h = 0 \\ h(a) = h(b) = 0 \end{cases}$$

with

$$L(h) = \frac{d}{dx}\left(p(x)\frac{dh}{dx}\right) + q(x)h$$

Show that the eigenfunctions are real, i.e. show that $\lambda = \overline{\lambda}$. HINT: You will need Lagrange's identity, $\int_a^b (uL(v) - vL(u))dx = p(x)\left(u\frac{dv}{dx} - v\frac{du}{dx}\right)\Big|_a^b$ and consider the two Sturm-Liouville problems for λ associated to the solution $\varphi(x)$ and $\overline{\lambda}$ associated to the solution $\overline{\varphi}(x)$.

7. Solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} u_t = k\Delta u \\ u(0, y, t) = u(L, y, t) = 0 \\ u(x, 0, t) = u(x, H, t) = 0 \\ u(x, y, 0) = \alpha(x, y) \end{cases}$$

where $k > 0$ is a constant. You do not need to derive what the coefficients are in the resulting series solution, just state the formulas for them if you can. You do not have to do the analysis for the underlying boundary value problems just state the eigenvalues and eigenfunctions.

8. Solve the following PDE subject to the following boundary and initial conditions:

$$\begin{cases} u_t = k\Delta u \\ u(0, y, t) = u(L, y, t) = 0 \\ u_y(x, 0, t) = u_y(x, H, t) = 0 \\ u(x, y, 0) = \alpha(x, y) \end{cases}$$

where $k > 0$ is a constant. You do not need to derive what the coefficients are in the resulting series solution, just state the formulas for them if you can. You do not have to do the analysis for the underlying boundary value problems just state the eigenvalues and eigenfunctions.

9. Let

$$u(x,t) = \frac{1}{2}\left(g(x+ct) + g(x-ct) + \int_{x-ct}^{x+ct} h(y)\ dy\right)$$

for some twice differentiable function g and once differentiable function h. Show that $u_{tt} = c^2 u_{xx}$ HINT: Just compute the derivatives and see what happens.