Ans Key

1.) $\begin{cases} u_t = \kappa u_{xx} \\ u_x(0,t) = u_x(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$

set $u(x,t) = h(x)g(t)$, then $u_t = h\frac{dg}{dt}$, and $u_{xx} = g\frac{d^2h}{dx^2}$ $\Rightarrow$ $h\frac{dg}{dt} = \kappa g \frac{d^2h}{dx^2}$

$\Rightarrow \underbrace{\frac{1}{\kappa}\frac{1}{g}\frac{dg}{dt}}_{\text{in } t} = \underbrace{\frac{1}{h}\frac{d^2h}{dx^2}}_{\text{in } x} = -\lambda$ $\Rightarrow$ (1) $\frac{dg}{dt} = -\kappa\lambda g$

(2) $\begin{cases} \frac{d^2h}{dx^2} + \lambda h = 0 \\ \frac{dh}{dx}(0) = \frac{dh}{dx}(L) = 0 \end{cases}$

(1) yields $g(t) = e^{-\kappa\lambda t}$

(2) $\frac{d^2h}{dx^2} + \lambda h = 0 \Rightarrow r^2 + \lambda = 0$ so $r = \pm\sqrt{-\lambda}$ 3 cases

(a) $\underline{\lambda = 0}$: so $h(x) = c_1 + c_2 x$, $\frac{dh}{dx} = c_2$ so $c_2 = \frac{dh}{dx}(0) = \frac{dh}{dx}(L) = 0 \Rightarrow c_2 = 0$

so $h(x) = c_1$

(b) $\underline{\lambda < 0}$: so set $s^2 = -\lambda \Rightarrow r = \pm s$ so $h(x) = c_1 e^{sx} + c_2 e^{-sx}$ and $\frac{dh}{dx} = s(c_1 e^{sx} - c_2 e^{-sx})$

$0 = \frac{dh}{dx}(0) = s(c_1 - c_2) \Rightarrow c_1 = c_2$ since $s \neq 0$ so $\frac{dh}{dx} = s c_1 (e^{sx} - e^{-sx})$

and $0 = \frac{dh}{dx}(L) = s c_1 (e^{sL} - e^{-sL})$, so want $c_1 \neq 0$ know $s \neq 0$ so $e^{sL} = e^{-sL} \Rightarrow s = 0$ X

$\Rightarrow c_1 = 0$ and $c_2 = 0$

(c) $\underline{\lambda > 0}$: then $h(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$ and $\frac{dh}{dx} = \sqrt{\lambda}\left(-c_1\sin(\sqrt{\lambda}x) + c_2\cos(\sqrt{\lambda}x)\right)$

$0 = \frac{dh}{dx}(0) = \sqrt{\lambda}c_2 \Rightarrow c_2 = 0$ as $\lambda \neq 0$ so $\frac{dh}{dx} = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}x)$

then $0 = \frac{dh}{dx}(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L)$ want $c_1 \neq 0$ know $\lambda \neq 0$ so $\Rightarrow \sin(\sqrt{\lambda}L) = 0$

$\Rightarrow \sqrt{\lambda}L = n\pi \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n \geq 1$ and $\varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right)$

thus $u(x,t) = \sum_{n=0}^{\infty} a_n e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right)$ $a_0 = \frac{1}{L}\int_0^L f(x)\,dx$, $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ $n \geq 1$

2) $\begin{cases} u_t = \kappa u_{xx} \\ u(-L,t) = u(L,t), \quad u_x(-L,t) = u_x(L,t) \\ u(x,0) = f(x) \end{cases}$

set $u(x,t) = h(x)g(t)$ so $u_t = h\frac{dg}{dt}$, $u_{xx} = g\frac{d^2h}{dx^2}$ $\Rightarrow$ $h\frac{dg}{dt} = \kappa g \frac{d^2h}{dx^2}$

$\Rightarrow \underbrace{\frac{1}{\kappa}\frac{1}{g}\frac{dg}{dt}}_{\text{in } t} = \underbrace{\frac{1}{h}\frac{d^2h}{dx^2}}_{\text{in } x} = -\lambda$ $\Rightarrow$ (1) $\frac{dg}{dt} = -\kappa\lambda g$ (2) $\begin{cases} \frac{d^2h}{dx^2} + \lambda h = 0 \\ h(-L) = h(L), \quad \frac{dh}{dx}(-L) = \frac{dh}{dx}(L) \end{cases}$

(1) yields $g(t) = e^{-\lambda\kappa t}$

(2) $\frac{d^2h}{dx^2} + \lambda h = 0 \Rightarrow r^2 + \lambda = 0$ so $r = \pm\sqrt{-\lambda}$ 3 cases

(a) $\underline{\lambda = 0}$: $h(x) = c_1 + c_2 x$ $\quad \frac{dh}{dx} = c_2$ so $\frac{dh}{dx}(-L) = \frac{dh}{dx}(L)$ automatically

the $h(L) = c_1 + c_2 L$
$h(-L) = c_1 - c_2 L$ so $h(L) = h(-L) \Rightarrow c_1 - c_2 L = c_1 + c_2 L \Rightarrow 2c_2 L = 0 \Rightarrow c_2 = 0$

so $h(x) = c_1$

(b) $\underline{\lambda < 0}$: set $s^2 = -\lambda$ so $r = \pm s$ so $h(x) = c_1 e^{sx} + c_2 e^{-sx}$ & $\frac{dh}{dx} = s(c_1 e^{sx} - c_2 e^{-sx})$

$h(-L) = c_1 e^{-sL} + c_2 e^{sL}$, $h(L) = c_1 e^{sL} + c_2 e^{-sL}$, $\frac{dh}{dx}(-L) = s(c_1 e^{-sL} - c_2 e^{sL})$, $\frac{dh}{dx}(L) = s(c_1 e^{sL} - c_2 e^{-sL})$

then the B.C. $\Rightarrow \begin{cases} c_1 e^{-sL} + c_2 e^{sL} = c_1 e^{sL} + c_2 e^{-sL} \\ s(c_1 e^{-sL} - c_2 e^{sL}) = s(c_1 e^{sL} - c_2 e^{-sL}) \end{cases} \Rightarrow \begin{cases} c_1 e^{-sL} + c_2 e^{sL} = c_1 e^{sL} + c_2 e^{-sL} \\ c_1 e^{-sL} - c_2 e^{sL} = c_1 e^{sL} - c_2 e^{-sL} \end{cases}$ as $s \neq 0$

adding the two gives $2c_1 e^{-sL} = 2c_1 e^{sL} \Rightarrow 2c_1(e^{-sL} - e^{sL}) = 0$ $\Rightarrow e^{-sL} = e^{sL} \Rightarrow s = 0$ X

so $c_1 = 0$, subtracting the two gives $2c_2 e^{sL} = 2c_2 e^{-sL}$ similarly $\Rightarrow c_2 = 0$

(c) $\underline{\lambda > 0}$ so $h(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$ & $\frac{dh}{dx} = \sqrt{\lambda}(-c_1 \sin(\sqrt{\lambda}x) + c_2 \cos(\sqrt{\lambda}x))$

$h(-L) = c_1\cos(\sqrt{\lambda}L) - c_2\sin(\sqrt{\lambda}L)$, $h(L) = c_1\cos(\sqrt{\lambda}L) + c_2\sin(\sqrt{\lambda}L)$

$\frac{dh}{dx}(-L) = \sqrt{\lambda}(c_1\sin(\sqrt{\lambda}L) + c_2\cos(\sqrt{\lambda}L))$ & $\frac{dh}{dx}(L) = \sqrt{\lambda}(-c_1\sin(\sqrt{\lambda}L) + c_2\cos(\sqrt{\lambda}L))$

$h(-L) = h(L) \Rightarrow 2c_2\sin(\sqrt{\lambda}L) = 0 \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n \geq 1$

$\frac{dh}{dx}(-L) = \frac{dh}{dx}(L) \Rightarrow 2c_1\sin(\sqrt{\lambda}L) = 0$ also yields so $\varphi_n(x) = \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)$

& $u(x,t) = \sum_{n=0}^{\infty} a_n e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right)$

$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$, $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$, $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

$n \geq 1$ $\qquad$ $n \geq 1$

3.) $\begin{cases} u_t = \kappa u_{xx} \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$

set $u(x,t) = h(x)g(t)$ so $u_t = h\frac{dg}{dt}$, $u_{xx} = g\frac{d^2h}{dx^2}$ $\Rightarrow$ $h\frac{dg}{dt} = \kappa g\frac{d^2h}{dx^2}$

$\Rightarrow \underbrace{\frac{1}{\kappa}\frac{1}{g}\frac{dg}{dt}}_{\text{in } t} = \underbrace{\frac{1}{h}\frac{d^2h}{dx^2}}_{\text{in } x} = -\lambda$ $\Rightarrow$ (1) $\frac{dg}{dt} = -\kappa\lambda g$ (2) $\begin{cases} \frac{d^2h}{dx^2} + \lambda h = 0 \\ h(0) = h(L) = 0 \end{cases}$

(1) yields $g(t) = e^{-\lambda\kappa t}$

(2) $\frac{d^2h}{dx^2} + \lambda h = 0$ $\Rightarrow$ $r^2 + \lambda = 0$ so $r = \pm\sqrt{-\lambda}$ 3 cases

(a) $\underline{\lambda = 0}$: so $h(x) = c_1 + c_2 x$ $0 = h(0) = c_1 \Rightarrow c_1 = 0$ so $h(x) = c_2 x$

and $0 = h(L) = c_2 L \Rightarrow c_2 = 0$

(b) $\underline{\lambda < 0}$: set $s^2 = -\lambda$ so $r = \pm s$ so $h(x) = c_1 e^{sx} + c_2 e^{-sx}$

$0 = h(0) = c_1 + c_2 \Rightarrow c_2 = -c_1$ so $h(x) = c_1(e^{sx} - e^{-sx})$

$0 = h(L) = c_1(e^{sL} - e^{-sL})$ but $c_1 \neq 0 \Rightarrow e^{sL} = e^{-sL} \Rightarrow s = 0$ but $s^2 = -\lambda \neq 0$ so $\times$

$\Rightarrow c_1 = 0$ and so $c_2 = 0$

(c) $\underline{\lambda > 0}$: so $h(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$ $0 = h(0) = c_1 \Rightarrow c_1 = 0$

so $h(x) = c_2\sin(\sqrt{\lambda}x)$ and $0 = h(L) = c_2\sin(\sqrt{\lambda}L)$ but $c_2 \neq 0$ so $\sin(\sqrt{\lambda}L) = 0$

$\Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n \geq 1$ and $\varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

thus $u(x,t) = \sum_{n=1}^{\infty} a_n e^{-\kappa\left(\frac{n\pi}{L}\right)^2 t}\sin\left(\frac{n\pi x}{L}\right)$, $a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ $n \geq 1$

4.) $\begin{cases} \Delta u = 0 \\ u(0,y) = u(L,y) = u(x,H) = 0 \\ u(x,0) = f(x) \end{cases}$

set $u(x,y) = h(x)g(y)$ so $u_{xx} = \frac{d^2h}{dx^2}g$, $u_{yy} = \frac{d^2g}{dy^2}h \Rightarrow \frac{d^2h}{dx^2}g + h\frac{d^2g}{dy^2} = 0$

$\Rightarrow \underbrace{-\frac{1}{h}\frac{d^2h}{dx^2}}_{\text{in } x} = \underbrace{\frac{1}{g}\frac{d^2g}{dy^2}}_{\text{in } y} = \lambda \Rightarrow (1) \begin{cases} \frac{d^2h}{dx^2} + \lambda h = 0 \\ h(0) = h(L) = 0 \end{cases}$ $(2) \begin{cases} \frac{d^2g}{dy^2} - \lambda g = 0 \\ g(H) = 0 \end{cases}$

for (1) $\frac{d^2h}{dx^2} + \lambda h = 0 \Rightarrow r^2 + \lambda = 0$ so $r = \pm\sqrt{-\lambda}$ 3 cases

(a) $\underline{\lambda = 0}$: $h(x) = c_1 + c_2 x$ $0 = h(0) = c_1 \Rightarrow c_1 = 0$ so $h(x) = c_2 x$ so $0 = h(L) = c_2 L$

$\Rightarrow c_2 = 0$

(b) $\underline{\lambda < 0}$: set $s^2 = -\lambda$ so $r = \pm s$ so $h(x) = c_1 e^{sx} + c_2 e^{-sx}$

$0 = h(0) = c_1 + c_2 \Rightarrow c_2 = -c_1$ thus $h(x) = c_1(e^{sx} - e^{-sx})$

$0 = h(L) = c_1(e^{sL} - e^{-sL}) \Rightarrow e^{sL} = e^{-sL} \Rightarrow s = 0$ but $s^2 = -\lambda \neq 0$ X

so $c_1 = 0 \Rightarrow c_2 = 0$

(c) $\underline{\lambda > 0}$: $h(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$ so $0 = h(0) = c_1 \Rightarrow h(x) = c_2\sin(\sqrt{\lambda}x)$

thus $0 = h(L) = c_2\sin(\sqrt{\lambda}L)$ want $c_2 \neq 0 \Rightarrow \lambda_n = \left(\frac{n\pi}{L}\right)^2$ $n \geq 1$ and $\varphi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$

so (2) now $\frac{d^2g}{dy^2} - \left(\frac{n\pi}{L}\right)^2 g = 0 \Rightarrow r^2 - \left(\frac{n\pi}{L}\right)^2 = 0 \Rightarrow r = \pm\frac{n\pi}{L}$ so $g_n(y) = c_1 e^{\frac{n\pi y}{L}} + c_2 e^{\frac{-n\pi y}{L}}$

$0 = g_n(H) = c_1 e^{\frac{n\pi H}{L}} + c_2 e^{\frac{-n\pi H}{L}} \Rightarrow c_2 e^{\frac{-n\pi H}{L}} = -c_1 e^{\frac{n\pi H}{L}} \Rightarrow c_1 = -c_2 e^{-\frac{2n\pi H}{L}}$

thus $g_n(y) = c_1\left(e^{\frac{n\pi y}{L}} - e^{\frac{-2n\pi H}{L}} e^{-\frac{n\pi y}{L}}\right) = c_1\sinh\left(\frac{n\pi}{L}(-H+y)\right)$

so $u(x,y) = \sum_{n=1}^{\infty} a_n \sinh\left(\frac{n\pi}{L}(-H+y)\right)\sin\left(\frac{n\pi x}{L}\right)$ and $a_n = \frac{2}{L}\frac{1}{\sinh\left(-\frac{n\pi}{L}H\right)}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

5.) $f(x) = \begin{cases} 0 & x < \frac{L}{2} \\ x & x > \frac{L}{2} \end{cases}$ $\quad$ & $f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\int_{\frac{L}{2}}^{L} x\,dx = \frac{1}{2L}\frac{x^2}{2}\Big|_{\frac{L}{2}}^{L} = \frac{1}{4L}\left(L^2 - \frac{L^2}{4}\right) = \frac{3L}{16}$$

$$\underset{n\geq 1}{a_n} = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{\frac{L}{2}}^{L} x\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left(\frac{L}{n\pi} x\sin\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{2}}^{L} - \frac{L}{n\pi}\int_{\frac{L}{2}}^{L}\sin\left(\frac{n\pi x}{L}\right)dx\right)$$

$u = x \quad dv = \cos\left(\frac{n\pi x}{L}\right)dx$

$du = dx \quad v = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$

$$= \frac{1}{L}\left(-\frac{L}{n\pi}\frac{L}{2}\sin\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{2}}^{L}\right) = \frac{1}{L}\left(\frac{-L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{L^2}{n^2\pi^2}(-1)^n - \frac{L^2}{n^2\pi^2}\cos\left(\frac{n\pi}{2}\right)\right)$$

$$\underset{n\geq 1}{b_n} = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{\frac{L}{2}}^{L} x\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left(-\frac{L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{2}}^{L} + \frac{L}{n\pi}\int_{\frac{L}{2}}^{L}\cos\left(\frac{n\pi x}{L}\right)dx\right)$$

$$= \frac{1}{L}\left(-\frac{L^2}{n\pi}(-1)^n + \frac{L^2}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right)$$

6.) $f(x) = \begin{cases} -1 & x < \frac{L}{2} \\ 1 & x > \frac{L}{2} \end{cases}$ $\quad$ & $f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2L}\left(-\int_{-L}^{\frac{L}{2}}dx + \int_{\frac{L}{2}}^{L}dx\right) = \frac{1}{2L}\left(-\frac{L}{2} - L + L - \frac{L}{2}\right) = \frac{1}{2L}(-L) = -\frac{1}{2}$$

$$\underset{n\geq 1}{a_n} = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left(-\int_{-L}^{\frac{L}{2}}\cos\left(\frac{n\pi x}{L}\right)dx + \int_{\frac{L}{2}}^{L}\cos\left(\frac{n\pi x}{L}\right)dx\right)$$

$$= \frac{1}{L}\left(-\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{\frac{L}{2}} + \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{2}}^{L}\right) = \frac{1}{n\pi}\left(-\sin\left(\frac{n\pi}{2}\right) - \sin\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{-2}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$\underset{n\geq 1}{b_n} = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\left(-\int_{-L}^{\frac{L}{2}}\sin\left(\frac{n\pi x}{L}\right)dx + \int_{\frac{L}{2}}^{L}\sin\left(\frac{n\pi x}{L}\right)dx\right)$$

$$= \frac{1}{n\pi}\left(\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{\frac{L}{2}} - \cos\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{2}}^{L}\right) = \frac{1}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) - (-1)^n - (-1)^n + \cos\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{2}{n\pi}\left(-(-1)^n + \cos\left(\frac{n\pi}{2}\right)\right)$$

7.) $\begin{cases} u_t = \kappa u_{xx} \\ u(-L,t) = T_1(t), \quad u_x(L,t) = T_2(t) \quad (*) \\ u(x,0) = f(x) \end{cases}$

let $w = u_1 - u_2$, where u_1 and u_2 both solve $(*)$ and $u_1 \neq u_2$

then $w_t = \kappa w_{xx}$ and $w(-L,t) = T_1(t) - T_1(t) = 0$, $w_x(L,t) = T_2(t) - T_2(t) = 0$

and $w(x,0) = f(x) - f(x) = 0$ so w solves $(**)$ $\begin{cases} w_t = \kappa w_{xx} \\ w(-L,t) = w_x(L,t) = 0 \\ w(x,0) = 0 \end{cases}$

consider $E(t) = \int_{-L}^{L} w^2\,dx$ then since $w^2 \geq 0$ so is $E(t)$ i.e. $E(t) \geq 0$

also $E(0) = \int_{-L}^{L} w^2(x,0)\,dx = \int_{-L}^{L} 0\,dx = 0$. Then $E'(t) = \int_{-L}^{L} 2 w w_t\,dx$ using $(**)$

get $2\int_{-L}^{L} w w_t\,dx = 2\kappa \int_{-L}^{L} w w_{xx}\,dx = 2\kappa\left(w w_x \Big|_{-L}^{L} - \int_{-L}^{L} w_x^2\,dx\right)$ by using by parts

but $w(-L,t) = w_x(L,t) = 0$ so $w w_x \Big|_{-L}^{L} = 0 \Rightarrow E'(t) = -2\kappa \int_{-L}^{L} w_x^2\,dx \leq 0$

since $w_x^2 \geq 0$, $\kappa > 0$ so $E \searrow$ and w/ $E \geq 0$ and $E(0) = 0 \Rightarrow E \equiv 0$

$\Rightarrow w \equiv 0$ so $u_1 = u_2$ hence unique sol'n.

8.) $\begin{cases} u_t = \kappa u_{xx} \\ u(0,t) = T_1(t), \quad u(L,t) = T_2(t) \quad (*) \\ u(x,0) = f(x) \end{cases}$

let u_1, u_2 both solve $(*)$ and $u_1 \neq u_2$. Set $w = u_1 - u_2$ then w solves

$(**) \begin{cases} w_t = \kappa w_{xx} \\ w(0,t) = w(L,t) = 0 \\ w(x,0) = 0 \end{cases}$ (similar to 7) consider $E(t) = \int_0^L w^2\,dx$ then $w^2 \geq 0$

so $E(t) \geq 0$. Next $E(0) = \int_0^L w^2(x,0)\,dx = \int_0^L 0\,dx = 0$ Then using $(**)$

$E'(t) = 2\int_0^L w w_t\,dx = 2\kappa \underbrace{\int_0^L w w_{xx}\,dx = 2\kappa\left(w w_x\Big|_0^L - \int_0^L w_x^2\,dx \right)}_{\text{by parts}} = 2\kappa\left(\underset{\text{from B.C.}}{0} - \int_0^L w_x^2\,dx\right)$

so $E'(t) = -2\kappa \int_0^L w_x^2\,dx$ since $\kappa, w_x^2 \geq 0 \Rightarrow E'(t) \leq 0$

so $E \searrow$, $E(0) = 0$ and $E \geq 0 \Rightarrow E \equiv 0 \Rightarrow w \equiv 0$ so $u_1 = u_2$ hence unique sol'n.

9.) $$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} f(y)\, e^{-\frac{(x-y)^2}{4t}}\, dy = (4\pi t)^{-\frac{1}{2}} \int_{\mathbb{R}} f(y)\, e^{-\frac{(x-y)^2}{4t}}\, dy$$

Then $$u_t = -\tfrac{1}{2}(4\pi t)^{-\frac{3}{2}} 4\pi \int_{\mathbb{R}} f(y)\, e^{-\frac{(x-y)^2}{4t}}\, dy + (4\pi t)^{-\frac{1}{2}} \int_{\mathbb{R}} f(y)\, e^{-\frac{(x-y)^2}{4t}} \frac{(x-y)^2}{4t^2}\, dy = (1) + (2)$$

$$u_x = (4\pi t)^{-\frac{1}{2}} \int_{\mathbb{R}} f(y)\, e^{-\frac{(x-y)^2}{4t}} \left(\frac{-2(x-y)}{4t}\right) dy$$

$$u_{xx} = (4\pi t)^{-\frac{1}{2}} \int_{\mathbb{R}} f(y)\, e^{-\frac{(x-y)^2}{4t}} \frac{4(x-y)^2}{16t^2}\, dy + (4\pi t)^{-\frac{1}{2}} \int_{\mathbb{R}} f(y)\, e^{-\frac{(x-y)^2}{4t}} \left(\frac{-2}{4t}\right) dy$$

$$= (3) + (4)$$

can see $(2) = (3)$ and rewriting $(4\pi t)^{-\frac{3}{2}} = \frac{1}{4t}\frac{1}{(4\pi t)^{\frac{1}{2}}}$ in (1) can see $(1) = (4)$

so $u_t = u_{xx}$