Name:____________________________

Math 3333 Section 002 *Section 8.2 Handout* *April 20, 2015*

Follow the instructions for each question and show enough of your work so that I can follow your thought process.

1. Let V be a vector space and $L : V \to V$ be a self-adjoint linear transformation so that it has the following spectral decomposition:

$$L = \int_{\mathbb{R}} \lambda \; dP_E(\lambda)$$

Suppose $f(x)$ is differentiable on all of $\mathbb{R}$. Using the functional calculus, show that:

$$f'(L) = \int_{\mathbb{R}} f'(\lambda) \; dP_E(\lambda)$$

(HINT: Use the definition of the derivative.)

2. Let V be a vector space and $L : V \to V$ be a self-adjoint linear transformation so that it has the following spectral decomposition:

$$L = \int_{\mathbb{R}} \lambda \; dP_E(\lambda)$$

Suppose $f(x)$ is continuous on all of $\mathbb{R}$. Using the functional calculus, show that for any $a,b \in \mathbb{R}$:

$$\int_a^b f(tL) \; dt = \int_{\mathbb{R}} \int_a^b f(t\lambda) \; dt \; dP_E(\lambda)$$

(HINT: Use the definition of the integral.)

3. Let A be an $n \times n$ symmetric matrix with spectral decomposition:

$$A = \sum_{i=1}^{n} \lambda_i P_E(\lambda_i)$$

Let

$$p_A(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

be the characteristic polynomial for A. Show that

$$A^n + a_1 A^{n-1} + \cdots a_{n-1}A + a_n I_n = 0$$

Moreover is A is nonsingular, show that

$$A^{-1} = -\frac{1}{a_n}(A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-2}A + a_{n-1}I_n)$$

(HINT: Use the functional calculus and the fact that a polynomial is continuous.)

4. Let $V = S_{nn}(\mathbb{R})$, the inner product space of all symmetric matrices, with inner product defined as

$$\langle A, B\rangle = \operatorname{tr}(B^t A)$$

In this space it is known that $\|A\| < \infty$ for all $A \in V$. Let $f(x)$ be a continuous function on the closed interval $[a, b]$ and let A have the following spectral decomposition:

$$A = \sum_{i=1}^{n} \lambda_i P_E(\lambda_i)$$

Show that $\|f(A)\| < \infty$ for all $A \in V$ (HINT: Use the functional calculus and the fact that a continuous function on a closed interval is a bounded function, i.e. it never has vertical asymptote.)