Practice Final Exam Solutions

1.) Consider $(A^t)^2 = A^t A^t = (AA)^t = (A^2)^t = A^t$ since A is idempotent.

Now consider $(A+B)^2 = A^2 + AB + BA + B^2 = A + B + AB + BA \neq A + B$ in general.

So $A+B$ is not idempotent in general.

2.) Recall $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. Using this consider formally

$(I_n - A)^{-1} = \frac{1}{I_n - A} = \sum_{i=0}^{\infty} A^i$ but $A^k = 0$ for some k $\Rightarrow \sum_{i=0}^{\infty} A^i = \sum_{i=0}^{k-1} A^i$ as all powers after k of A equal the zero matrix

so finite sum, always converges. Thus $(I_n - A)^{-1} = \sum_{i=0}^{k-1} A^i = I_n + A + \cdots + A^{k-1}$

3.) have $S = \{-t^2 + t + 2,\ 2t^2 + 2t + 3,\ 4t^2 - 1\}$ notice S has 3 vectors spanning a dim $P_2 = 3$. So S could be a basis. Just have to check if linear indep.

consider $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \Rightarrow c_1(-t^2 + t + 2) + c_2(2t^2 + 2t + 3) + c_3(4t^2 - 1) = 0$

$\Rightarrow (-c_1 + 2c_2 + 4c_3)t^2 + (c_1 + 2c_2)t + (2c_1 + 3c_2 - c_3)\cdot 1 = 0$

$$\Rightarrow \begin{cases} -c_1 + 2c_2 + 4c_3 = 0 \\ c_1 + 2c_2 = 0 \\ 2c_1 + 3c_2 - c_3 = 0 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} -1 & 2 & 4 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 3 & -1 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\Rightarrow S$ is dep. so not a basis for P_2

4.) have $S = \{t^2 + 1,\ 3t^2 + 2t + 1,\ 6t^2 + 6t + 3\}$: S has 3 vectors spanning a dim $P_2 = 3$, like (3) just check if S is lin. indep.

so $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0 \Rightarrow c_1(t^2 + 1) + c_2(3t^2 + 2t + 1) + c_3(6t^2 + 6t + 3) = 0$

$\Rightarrow (c_1 + 3c_2 + 6c_3)t^2 + (2c_2 + 6c_3)t + (c_1 + c_2 + 3c_3)\cdot 1 = 0$

$$\Rightarrow \begin{cases} c_1 + 3c_2 + 6c_3 = 0 \\ 2c_2 + 6c_3 = 0 \\ c_1 + c_2 + 3c_3 = 0 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1 & 3 & 6 & 0 \\ 0 & 2 & 6 & 0 \\ 1 & 1 & 3 & 0 \end{array}\right) \xrightarrow{\text{rref}} I_3$$

$\Rightarrow S$ is lin indep. So S is a basis for P_2

5.) $S = \{1, \sin x, \cos x\}$ we compute L on basis vectors.

$L(1) = 0 = 0(1) + 0(\sin x) + 0(\cos x)$, $L(\sin x) = \cos x = 0(1) + 0(\sin x) + 1(\cos x)$

and $L(\cos x) = -\sin x = 0(1) + (-1)(\sin x) + 0(\cos x)$

so $L(1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $L(\sin x) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $L(\cos x) = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ $\Rightarrow$ $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

6.) $S = \{1, \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\}$ we compute L on basis vectors

$L(1) = 0 = 0(1) + 0\left(\sin\left(\frac{n\pi x}{L}\right)\right) + 0\left(\cos\left(\frac{n\pi x}{L}\right)\right)$, $L\left(\sin\left(\frac{n\pi x}{L}\right)\right) = -\frac{n^2\pi^2}{L^2}\sin\left(\frac{n\pi x}{L}\right) = 0(1) - \frac{n^2\pi^2}{L^2}\left(\sin\left(\frac{n\pi x}{L}\right)\right) + 0\left(\cos\left(\frac{n\pi x}{L}\right)\right)$

$L\left(\cos\left(\frac{n\pi x}{L}\right)\right) = -\frac{n^2\pi^2}{L^2}\cos\left(\frac{n\pi x}{L}\right) = 0(1) + 0\left(\sin\left(\frac{n\pi x}{L}\right)\right) + -\frac{n^2\pi^2}{L^2}\left(\cos\left(\frac{n\pi x}{L}\right)\right)$

so $L(1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $L\left(\sin\left(\frac{n\pi x}{L}\right)\right) = \begin{pmatrix} 0 \\ -\frac{n^2\pi^2}{L^2} \\ 0 \end{pmatrix}$, $L\left(\cos\left(\frac{n\pi x}{L}\right)\right) = \begin{pmatrix} 0 \\ 0 \\ -\frac{n^2\pi^2}{L^2} \end{pmatrix}$

so $A = -\frac{n^2\pi^2}{L^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

7.) $A = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix}$ so $p_A(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} -\lambda & 0 & 3 \\ 1 & -\lambda & -1 \\ 0 & 1 & 3-\lambda \end{vmatrix} = \lambda^3 - 3\lambda^2 + \lambda - 3 = (\lambda-3)(\lambda^2+1)$

so the eigenvalues are $\lambda = 3, i, -i$

$\underline{\lambda = 3}$: $(A - 3I_3)x = 0 \Rightarrow \left(\begin{array}{ccc|c} -3 & 0 & 3 & 0 \\ 1 & -3 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\underline{\lambda = i}$: $(A - iI_3)x = 0 \Rightarrow \left(\begin{array}{ccc|c} -i & 0 & 3 & 0 \\ 1 & -i & -1 & 0 \\ 0 & 1 & 3-i & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & 3i & 0 \\ 0 & 1 & 3-i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow v = \begin{pmatrix} -3i \\ i-3 \\ 1 \end{pmatrix}$

$\underline{\lambda = -i}$: since $-i = \bar{i}$ i.e. conjugate to i we know the vector corresponding to $-i$ is $v = \overline{\begin{pmatrix} -3i \\ i-3 \\ 0 \end{pmatrix}} = \begin{pmatrix} 3i \\ -i-3 \\ 0 \end{pmatrix}$

8.) $A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{pmatrix}$ so $P_A(\lambda) = \det(A - \lambda I_3) = \begin{vmatrix} 2-\lambda & 1 & 2 \\ 2 & 2-\lambda & -2 \\ 3 & 1 & 1-\lambda \end{vmatrix} = (2-\lambda)(\lambda+1)(\lambda-4)$

so the eigenvalues are $\lambda = -1, 2, 4$

$\underline{\lambda = -1}$: $(A + I_3)x = 0 \Rightarrow \begin{pmatrix} 3 & 1 & 2 \\ 2 & 3 & -2 \\ 3 & 1 & 2 \end{pmatrix} \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & \frac{24}{21} & 0 \\ 0 & 1 & -\frac{10}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow v = \begin{pmatrix} -24 \\ 30 \\ 1 \end{pmatrix}$

$\underline{\lambda = 2}$: $(A - 2I_3)x = 0 \Rightarrow \left(\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 2 & 0 & -2 & 0 \\ 3 & 1 & -1 & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow v = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$\underline{\lambda = 4}$: $(A - 4I_3)x = 0 \Rightarrow \left(\begin{array}{ccc|c} -2 & 1 & 2 & 0 \\ 2 & -2 & -2 & 0 \\ 3 & 1 & -3 & 0 \end{array}\right) \xrightarrow{rref} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow v = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

9.) 1st $P_A(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = (a-\lambda)(d-\lambda) - bc = ad - (a+d)\lambda - bc + \lambda^2$

$= \lambda^2 - (a+d)\lambda + ad - bc$ so $\lambda_{1,2} = \dfrac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

then $\lambda_1 \lambda_2 = \left(\dfrac{(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}}{2}\right)\left(\dfrac{(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}}{2}\right) = \frac{1}{4}\left((a+d)^2 - (a+d)^2 + 4(ad-bc)\right)$

$= ad - bc = \det(A)$.

10.) from (9) saw $P_A(\lambda) = \lambda^2 - (a+d)\lambda + ad - bc$ and $\lambda_{1,2} = \dfrac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

so $\lambda_1 + \lambda_2 = \dfrac{(a+d) + \sqrt{(a+d)^2 - 4(ad-bc)}}{2} + \dfrac{(a+d) - \sqrt{(a+d)^2 - 4(ad-bc)}}{2} = a + d = tr(A)$

Id & the lin. transform identity

11.) 1st $f'(L) = \lim_{h \to 0} \dfrac{f(L+h) - f(L)}{h} = \lim_{h \to 0} \frac{1}{h}\left(\int_{\mathbb{R}} f(\lambda+h)\, dP_E(\lambda) - \int_{\mathbb{R}} f(\lambda)\, dP_E(\lambda)\right)$

$= \lim_{h \to 0} \int_{\mathbb{R}} \dfrac{f(\lambda+h) - f(\lambda)}{h}\, dP_E(\lambda)$ by the decomp. and the fnc. calculus

so $f'(L) = \lim_{h \to 0} \int_{\mathbb{R}} \dfrac{f(\lambda+h) - f(\lambda)}{h}\, dP_E(\lambda) = \int_{\mathbb{R}} \lim_{h \to 0} \dfrac{f(\lambda+h) - f(\lambda)}{h}\, dP_E(\lambda) = \int_{\mathbb{R}} f'(\lambda)\, dP_E(\lambda)$

12.) 1st note $\int_a^b f(tL)\,dt = \lim_{n\to\infty} \sum_{i=0}^{n} f(t_i^* L)\,\Delta t$ where $[a,b]$ is subdivided into intervals $a = t_0 < t_1 < t_2 < \cdots < t_n = b$ and $\Delta t = \frac{b-a}{n}$, Then by the spectral decomp and fnc. calculus have

$$\lim_{n\to\infty} \sum_{i=0}^{n} f(t_i^* L)\,\Delta t = \lim_{n\to\infty} \sum_{i=0}^{n} \int_{\mathbb{R}} f(t_i^* \lambda)\, dP_E(\lambda)\,\Delta t = \lim_{n\to\infty} \int_{\mathbb{R}} \sum_{i=0}^{n} f(t_i^* \lambda)\,\Delta t\, dP_E(\lambda)$$

$$= \int_{\mathbb{R}} \lim_{n\to\infty} \sum_{i=0}^{n} f(t_i^* \lambda)\,\Delta t\, dP_E(\lambda) = \int_{\mathbb{R}} \int_a^b f(t\lambda)\,dt\, dP_E(\lambda)$$

13.) 1st notice $A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n I_n = P_A(A)$. But by the spectral decomp and fnc. calculus $P_A(A) = P_A\left(\sum_{i=1}^{k} \lambda_i P_E(\lambda_i)\right) = \sum_{i=1}^{k} P_A(\lambda_i) P_E(\lambda_i)$, but λ_i's are the eigenvalues to A so $P_A(\lambda_i) = 0$ for all $i = 1, \dots, k$ $\Rightarrow \sum_{i=1}^{k} P_A(\lambda_i) P_E(\lambda_i) = 0$

$\Rightarrow P_A(A) = 0$ so $A^n + a_1 A^{n-1} + \cdots + a_{n-1} A + a_n I_n = 0$

then $-a_n I_n = A^n + a_1 A^{n-1} + \cdots + a_{n-1} A = A(A^{n-1} + a_1 A^{n-2} + \cdots a_{n-1} I_n)$

mult. by A^{-1} on both sides $\Rightarrow -a_n A^{-1} = A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-1} I_n$

$\Rightarrow A^{-1} = \frac{-1}{a_n}\left(A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-1} I_n\right)$.