Name:______________________________

Math 4163 Section 001 *Practice Final Exam* *April 20, 2015*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work or answer, you will receive little or no credit!

1. Let A be an $n \times n$ matrix. Suppose that A has the following property: $A^2 = A$. We call this property, idempotent. Show that A^t is idempotent. Is it true that if both A and B are idempotent, then so it $A + B$?

2. Let A be an $n \times n$ matrix. Suppose that A has the following property: $A^k = 0$ for some $k \geq 1$. We call this property, nilpotent. Show that $I_n - A$ is an invertible matrix. (HINT: The inverse can be explictly computed, think of power series.)

3. Let

$$S = \{-t^2 + t + 2, 2t^2 + 2t + 3, 4t^2 - 1\}$$

Determine if S is a basis for P_2, the vector space of all polynomials of degree 2 or less.

4. Let

$$S = \{t^2 + 1, 3t^2 + 2t + 1, 6t^2 + 6t + 3\}$$

Determine if S is a basis for P_2, the vector space of all polynomials of degree 2 or less.

5. Let V be a subspace of $C^1(\mathbb{R})$ with basis $S = \{1, \sin x, \cos x\}$. Let $L : V \to V$ be a linear transformation defined by:

$$L(f) = \frac{df}{dx}$$

Compute the matrix representation of L with respect to the basis S.

6. Let V be a subspace of $C^1(\mathbb{R})$ with basis

$$S = \left\{1, \sin\left(\frac{n\pi x}{L}\right), \cos\left(\frac{n\pi x}{L}\right)\right\}.$$

Let $L : V \to V$ be a linear transformation defined by:

$$L(f) = \frac{d^2 f}{dx^2}$$

Compute the matrix representation of L with respect to the basis S.

7. Compute the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix}$$

8. Compute the eigenvalues and eigenvectors of the following matrix:

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & 2 & -2 \\ 3 & 1 & 1 \end{pmatrix}$$

9. Let A be a 2×2 matrix, that is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let λ_1 and λ_2 be the roots to $p_A(\lambda)$, the characteristic polynomial of A. Show that $\lambda_1 \lambda_2 = \det(A)$.

10. Let A be a 2×2 matrix, that is

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Let λ_1 and λ_2 be the roots to $p_A(\lambda)$, the characteristic polynomial of A. Show that $\lambda_1 + \lambda_2 = \operatorname{tr}(A)$.

11. Let V be a vector space and $L : V \to V$ be a self-adjoint linear transformation so that it has the following spectral decomposition:

$$L = \int_{\mathbb{R}} \lambda \; dP_E(\lambda)$$

Suppose $f(x)$ is differentiable on all of $\mathbb{R}$. Using the functional calculus, show that:

$$f'(L) = \int_{\mathbb{R}} f'(\lambda) \; dP_E(\lambda)$$

(HINT: Use the definition of the derivative.)

12. Let V be a vector space and $L : V \to V$ be a self-adjoint linear transformation so that it has the following spectral decomposition:

$$L = \int_{\mathbb{R}} \lambda \; dP_E(\lambda)$$

Suppose $f(x)$ is continuous on all of $\mathbb{R}$. Using the functional calculus, show that for any $a{,}b \in \mathbb{R}$:

$$\int_a^b f(tL) \; dt = \int_{\mathbb{R}} \int_a^b f(t\lambda) \; dt \; dP_E(\lambda)$$

(HINT: Use the definition of the integral.)

13. Let A be an $n \times n$ symmetric matrix with spectral decomposition:

$$A = \sum_{i=1}^{n} \lambda_i P_E(\lambda_i)$$

Let

$$p_A(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n$$

be the characteristic polynomial for A. Show that

$$A^n + a_1 A^{n-1} + \cdots a_{n-1}A + a_n I_n = 0$$

Moreover is A is nonsingular, show that

$$A^{-1} = -\frac{1}{a_n}(A^{n-1} + a_1 A^{n-2} + \cdots + a_{n-2}A + a_{n-1}I_n)$$

(HINT: Use the functional calculus and the fact that a polynomial is continuous.)