Practice Exam 3 Solutions

1.) $V = \mathbb{R}^2 \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \qquad \langle x, y \rangle = x_1 y_1 - x_2 y_1 - x_1 y_2 + 3 x_2 y_2$

$\langle x, x \rangle = x_1^2 - x_2 x_1 - x_1 x_2 + 3x_2^2 = x_1^2 - 2x_1 x_2 + 3x_2^2 = x_1^2 - 2x_1 x_2 + x_2^2 + 2x_2^2$

$= (x_1 - x_2)^2 + 2x_2^2 \geq 0$

$\langle x, y \rangle = x_1 y_1 - x_2 y_1 - x_1 y_2 + 3x_2 y_2 = y_1 x_1 - x_1 y_2 - x_2 y_1 + 3 y_2 x_2$

$= y_1 x_1 - y_2 x_1 - y_1 x_2 + 3 y_2 x_2 = \langle y, x \rangle$

$\langle cx, y \rangle = c x_1 y_1 - c x_2 y_1 - c x_1 y_2 + 3c x_2 y_2 = c\,(x_1 y_1 - x_2 y_1 - x_1 y_2 + 3 x_2 y_2)$

$= c \langle x, y \rangle$

$\langle x, y + z \rangle = x_1 (y_1 + z_1) - x_2 (y_1 + z_1) - x_1 (y_2 + z_2) + 3 x_2 (y_2 + z_2)$

$= x_1 y_1 - x_2 y_1 - x_1 y_2 + 3 x_2 y_2 + x_1 z_1 - x_2 z_1 - x_1 z_2 + 3 x_2 z_2$

$= \langle x, y \rangle + \langle x, z \rangle$ all four prop. satisfied so inner product

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2.) $V = C[-\pi, \pi] \qquad \langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) g(t)\, dt$

$\langle f, f \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)^2\, dt \geq 0$ since $f(t)^2 \geq 0$

$\langle f, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) g(t)\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(t) f(t)\, dt = \langle g, f \rangle$

$\langle cf, g \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} c f(t) g(t)\, dt = c \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) g(t)\, dt = c \langle f, g \rangle$

$\langle f, g + h \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) (g(t) + h(t))\, dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} (f(t) g(t) + f(t) h(t))\, dt$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) g(t)\, dt + \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) h(t)\, dt = \langle f, g \rangle + \langle f, h \rangle$

3.) $S = \{1, t\}$ set $v_1 = 1$, $v_2 = t$

$u_1 = v_1$ then $u_2 = v_2 - \text{Proj}_{u_1} v_2 = t - \dfrac{\langle u_1, v_2\rangle}{\|u_1\|^2} u_1$

so $\langle u_1, v_2\rangle = \int_0^1 1 \cdot t\,dt = \frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$ so $u_2 = t - \frac{\frac{1}{2}}{1} 1 = t - \frac{1}{2}$

$\|u_1\|^2 = \int_0^1 1^2 dt = t\Big|_0^1 = 1$

$w_1 = \frac{u_1}{\|u_1\|} = 1$, $w_2 = \frac{u_2}{\|u_2\|}$, $\|t - \frac{1}{2}\|^2 = \int_0^1 (t - \frac{1}{2})^2 dt$

$= \int_0^1 (t^2 - t + \frac{1}{4}) dt = \frac{t^3}{3} - \frac{t^2}{2} + \frac{1}{4} t\Big|_0^1 = \frac{1}{3} - \frac{1}{2} + \frac{1}{4}$
$= \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

so $\|u_2\| = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}}$

then $w_1 = 1$ and $w_2 = 2\sqrt{3}(t - \frac{1}{2})$

4.) $S = \{1, t, e^t\}$ $v_1 = 1$, $v_2 = t$, $v_3 = e^t$

$u_1 = v_1$ then $u_2 = v_2 - \text{Proj}_{u_1} v_2 = t - \dfrac{\langle u_1, v_2\rangle}{\|u_1\|^2} u_1 = t - \frac{1}{2}$ from last problem

then $u_3 = v_3 - \text{Proj}_{u_1} v_3 - \text{Proj}_{u_2} v_3 = e^t - \dfrac{\langle 1, e^t\rangle}{\|1\|^2} \cdot 1 - \dfrac{\langle t - \frac{1}{2}, e^t\rangle}{\|t - \frac{1}{2}\|^2}(t - \frac{1}{2})$

$\langle 1, e^t\rangle = \int_0^1 e^t dt = e - 1$, $\|1\|^2 = 1$

$\langle t - \frac{1}{2}, e^t\rangle = \int_0^1 (t - \frac{1}{2}) e^t dt = \int_0^1 te^t dt - \frac{1}{2} e^t\Big|_0^1 = te^t\Big|_0^1 - \int_0^1 e^t dt - \frac{1}{2}(e-1)$

$= e - (e-1) - \frac{1}{2}(e-1) = e - \frac{3}{2}(e-1) = \frac{3}{2} - \frac{1}{2} e$

$\|t - \frac{1}{2}\|^2 = \frac{1}{12}$ from previous problem.

so $u_3 = e^t - (e-1) - (\frac{3}{2} - \frac{1}{2} e)(t - \frac{1}{2}) = e^t - (\frac{3}{2} - \frac{1}{2} e) t + \frac{7}{4} - \frac{5}{4} e$

$w_1 = \frac{u_1}{\|u_1\|}$, $w_2 = \frac{u_2}{\|u_2\|}$, $w_3 = \frac{u_3}{\|u_3\|}$ $\|u_2\| = \frac{1}{2\sqrt{3}}$ from last problem

$\|u_3\|^2 = \int_0^1 \Big(e^t - \underbrace{(\tfrac{3}{2} - \tfrac{1}{2} e)}_{b} t + \underbrace{\tfrac{7}{4} - \tfrac{5}{4} e}_{c}\Big)^2 dt = \int_0^1 (e^t - bt + c)^2 dt = \int_0^1 (e^{2t} - 2bte^t + 2ce^t - 2bct + b^2t^2 + c^2) dt$

$= \frac{1}{2} e^{2t}\Big|_0^1 - 2bte^t\Big|_0^1 + 2be^t\Big|_0^1 + 2ce^t\Big|_0^1 - bct^2\Big|_0^1 + \frac{b^2 t^3}{3}\Big|_0^1 + c^2 t\Big|_0^1 = \frac{1}{2}(e^2 - 1) - 2b e + 2be - 2b + 2ce - 2c - bc + \frac{b^2}{3} + c^2$

$= \frac{1}{2}(e^2 - 1) - 2b + 2ce - bc + \frac{b^3}{3} + c^2 = \alpha$ so $\|u_3\| = \sqrt{\alpha}$

then $w_1 = 1$, $w_2 = 2\sqrt{3}(t - \frac{1}{2})$, $w_3 = \frac{1}{\sqrt{\alpha}}\left(e^t - (\frac{3}{2} - \frac{1}{2} e) t + \frac{7}{4} - \frac{5}{4} e\right)$

5.) $\det(I_n - AB) = \det(A(A^{-1} - B)) = \det(A)\det(A^{-1} - B)$

$= \det(A^{-1} - B)\det(A) = \det((A^{-1} - B)A) = \det(I_n - BA)$

6.) recall skew-symmetric $A^t = -A$ so $\det(A^t) = \det(\underset{\uparrow}{-}A) = (-1)^n \det(A)$

each row multiplied by -1

then $\det(A^t) = \det(A) \Rightarrow \det(A) = (-1)^n \det(A)$ if n odd then $(-1)^n = -1$

$\Rightarrow \det(A) = -\det(A) \Rightarrow \det(A) = 0$

7.) $L: P_3 \to \mathbb{R}$ $\quad p(t) = a_1 t^3 + a_2 t^2 + a_3 t + a_4$

So $L(p(t)) = \int_0^1 p(t)\,dt = \left(\frac{a_1 t^4}{4} + \frac{a_2 t^3}{3} + \frac{a_3 t^2}{2} + a_4 t\right)\Big|_0^1 = \frac{a_1}{4} + \frac{a_2}{3} + \frac{a_3}{2} + a_4$

now want $p(t)$ s.t. $L(p(t)) = 0 \Rightarrow \frac{a_1}{4} + \frac{a_2}{3} + \frac{a_3}{2} + a_4 = 0$

$\Rightarrow a_4 = -\frac{1}{4}a_1 - \frac{1}{3}a_2 - \frac{1}{2}a_3$ w/ a_1, a_2, a_3 free ~~variable~~ parameters

so $p(t) = a_1 t^3 + a_2 t^2 + a_3 t + -\frac{1}{4}a_1 - \frac{1}{3}a_2 - \frac{1}{2}a_3 = a_1\left(t^3 - \frac{1}{4}\right) + a_2\left(t^2 - \frac{1}{3}\right) + a_3\left(t - \frac{1}{2}\right)$

so basis for $\ker(L)$ is $S = \left\{t^3 - \frac{1}{4},\ t^2 - \frac{1}{3},\ t - \frac{1}{2}\right\}$ and $\dim \ker(L) = 3$

8.) $L: P_2 \to P_2$ $\quad p(t) = a_1 t^2 + a_2 t + a_3$

so $L(p(t)) = t\frac{dp}{dt} = t(2a_1 t + a_2) = 2a_1 t^2 + a_2 t$

Now want $p(t)$ s.t. $L(p(t)) = 0 \Rightarrow 2a_1 t^2 + a_2 t = 0$ ~~[illegible]~~

~~[illegible]~~ need to be zero for all t

so must have $a_1 = a_2 = 0$ so basis for $\ker(L)$ is $S = \{1\}$ & $\dim \ker(L) = 1$

Alt. way is to write basis for P_2 $S = \{1, t, t^2\}$ find associated matrix A for L

$L(1) = 0,\ L(t) = t,\ L(t^2) = 2t^2 \Rightarrow A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ clearly has dimension 1

9.) Since columns of P are orthonormal then $P^tP = I_n$ so

$\|Px\|^2 = \langle Px, Px\rangle = \langle P^tPx, x\rangle = \langle I_n x, x\rangle = \langle x, x\rangle = \|x\|^2$

so $\|Px\| = \|x\|$