# Practice Exam 2 Solutions

1.) $C^1(\mathbb{R}) = \{ f : f' \text{ exists} \}$

let $f, g \in C^1(\mathbb{R})$ then cond $(f+g)'(x) = f'(x) + g'(x)$ $f', g'$ exists

so $f+g \in C^1(\mathbb{R})$.

let $c$ scalar, $f \in C^1(\mathbb{R})$, then $(cf)'(x) = cf'(x)$ and $f'$ exists

so $cf \in C^1(\mathbb{R})$ so $C^1(\mathbb{R})$ is a subspace of $C(\mathbb{R})$

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2.) $C_0(\mathbb{R}) = \{ f : \lim\limits_{x \to \pm\infty} f(x) = 0 \}$

let $f, g \in C_0(\mathbb{R})$ cond $\lim\limits_{x \to \infty} (f+g)(x) = \lim\limits_{x\to\infty} f(x) + \lim\limits_{x\to\infty} g(x) = 0 + 0 = 0$

$\lim\limits_{x \to -\infty} (f+g)(x) = \lim\limits_{x\to-\infty} f(x) + \lim\limits_{x\to-\infty} g(x) = 0 + 0 = 0$

so $f+g \in C_0(\mathbb{R})$.

let $c$ scalar and $f \in C_0(\mathbb{R})$ cond $\lim\limits_{x\to\infty} cf(x) = c \lim\limits_{x\to\infty} f(x) = c \cdot 0 = 0$

and $\lim\limits_{x\to-\infty} cf(x) = c \lim\limits_{x\to-\infty} f(x) = c \cdot 0 = 0$

so $cf \in C_0(\mathbb{R})$, so $C_0(\mathbb{R})$ is a subspace of $C(\mathbb{R})$

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3.) $S = \{ t^2+1, t^2+t, t+1 \}$ and $V = P_2$

notice $\dim(V) = 3$ and $S$ has 3 vectors so we just ~~[illegible]~~ check lin. dep. for $S$. dep eq $\Rightarrow$ $c_1 v_1 + c_2 v_2 + c_3 v_3 = 0$

$\Rightarrow c_1 t^2 + c_1 + c_2 t^2 + c_2 t + c_3 t + c_3 = 0 \Rightarrow (c_1 + c_2) t^2 + (c_2 + c_3) t + (c_1 + c_3) = 0$

or $\begin{cases} c_1 + c_2 = 0 \\ c_2 + c_3 = 0 \\ c_1 + c_3 = 0 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$

so $S$ is lin. indep. and hence $S$ is a basis for $P_2$

4.) $S = \left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 2 \\ 8 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 5 \end{pmatrix} \right\}$ and $V = \mathbb{R}^4$

notice $\dim(V) = 4$ and $S$ has 4 vectors so we just check lin. dep. for $S$.

dep. eq. $\Rightarrow c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4 = 0 \Rightarrow \begin{pmatrix} 2 \\ -1 \\ 1 \\ 5 \end{pmatrix} c_1 + \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} c_2 + \begin{pmatrix} 3 \\ 0 \\ 2 \\ 8 \end{pmatrix} c_3 + \begin{pmatrix} 2 \\ 1 \\ 1 \\ 5 \end{pmatrix} c_4 = 0$

or $\left(\begin{array}{cccc|c} 2 & 1 & 3 & 2 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 2 & 1 & 0 \\ 5 & 1 & 8 & 5 & 0 \end{array}\right) \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & \frac{6}{5} \\ 0 & 0 & 0 & 0 \end{pmatrix}$ have a row of zeros

and 4 consts $c_1, c_2, c_3, c_4$ so ~~the~~ some are nonzero and hence lin dep.
so $S$ is not a basis for $V$.

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5.) study $Ax = 0$ gives $\left(\begin{array}{ccccc|c} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & -1 & 3 & 0 \\ 3 & -1 & 3 & 0 & 3 & 0 \\ 4 & -2 & 5 & 1 & 3 & 0 \\ 1 & 3 & -4 & -5 & 6 & 0 \end{array}\right)$

$\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -3 & 6 \\ 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ so
$x_1 = 0$
$x_2 - 3x_4 + 6x_5 = 0$
$x_3 - x_4 + 3x_5 = 0$
$x_4 = x_4$
$x_5 = x_5$
$\Rightarrow$
$x_1 = 0$
$x_2 = 3x_4 - 6x_5$
$x_3 = x_4 - 3x_5$
$x_4 = x_4$
$x_5 = x_5$

so $X = \begin{pmatrix} 0 \\ 3 \\ 1 \\ 1 \\ 0 \end{pmatrix} x_4 + \begin{pmatrix} 0 \\ -6 \\ -3 \\ 0 \\ 1 \end{pmatrix} x_5$ basis is $\left\{ \begin{pmatrix} 0 \\ 3 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -6 \\ -3 \\ 0 \\ 1 \end{pmatrix} \right\}$

and ~~dim~~ $\dim \ker(A) = 2$

6.) study $Ax=0$ gives $\left(\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 0 \\ 2 & 3 & 2 & 0 & 0 \\ 3 & 4 & 1 & 1 & 0 \\ 1 & 1 & -1 & 1 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cccc|c} 1 & 0 & -5 & 3 & 0 \\ 0 & 1 & 4 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

so $\begin{array}{l} x_1 - 5x_3 + 3x_4 = 0 \\ x_2 + 4x_3 - 2x_4 = 0 \\ x_3 = x_3 \\ x_4 = x_4 \end{array} \Rightarrow \begin{array}{l} x_1 = 5x_3 - 3x_4 \\ x_2 = -4x_3 + 2x_4 \\ x_3 = x_3 \\ x_4 = x_4 \end{array}$ so $x = \begin{pmatrix} 5 \\ -4 \\ 1 \\ 0 \end{pmatrix} x_3 + \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix} x_4$

basis is $\left\{ \begin{pmatrix} 5 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ and $\dim \ker(A) = 2$

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7.) 1st assume $Ax=0$ has only trivial soln. $\Rightarrow \dim\ker(A) = 0$ so by rank-nullity thm $\Rightarrow rk(A) = n$ so rows of $A$ lin. indp. $\Rightarrow$ columns of $A$ lin. indp.

2nd now assume columns of $A$ lin. indp. $\Rightarrow$ rows of $A$ lin. indp. $\Rightarrow rk(A) = n$ so by rank-nullity thm $\Rightarrow \dim\ker(A) = 0 \Rightarrow Ax=0$ has only trivial soln.

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8.) let $A$ $m\times n$, notice $A^t$ is $n \times m$ so $A^tA$ is $n \times n$ if $A^tA$ is nonsingular $\Rightarrow \dim\ker(A^tA) = 0$ so only soln to $A^tAx=0$ is trivial soln. so consider $A^t(Ax) = 0 \Rightarrow Ax = 0$ since $A^tAx=0$ has only trivial soln $\Rightarrow Ax=0$ always $\Rightarrow x=0$ only $\Rightarrow \dim\ker(A) = 0$ (for any matrix $A$) so by rank-nullity thm $\Rightarrow rk(A) = n$.

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9.) let $A$ be $n\times n$, consider $Ax = x \Rightarrow Ax - x = 0 \Rightarrow (A - I_n)x = 0$ since there is no nonzero $x$ s.t. $Ax = x$ is true $\Rightarrow \dim\ker(A - I_n) = 0$ so by rank-nullity thm $\Rightarrow rk(A - I_n) = n$ so this means $A - I_n$ is nonsingular.