Practice Exam 1 Solutions

1.) $A = \begin{pmatrix} 1 & 0 & 3 \\ 4 & 5 & 7 \\ 2 & 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 3 & 2 & 1 \\ 5 & 5 & 1 \\ 1 & 8 & 1 \end{pmatrix}$

$$AB = \begin{pmatrix} 1 & 0 & 3 \\ 4 & 5 & 7 \\ 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 & 1 \\ 5 & 5 & 1 \\ 1 & 8 & 1 \end{pmatrix} = \begin{pmatrix} 6 & 26 & 4 \\ 34 & 87 & 16 \\ 15 & 41 & 7 \end{pmatrix}$$

& $tr(AB) = 6+87+7 = 100$

2.) $A = \begin{pmatrix} 3 & 0 & 2 \\ 8 & -1 & 4 \\ 2 & 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 4 & 5 \\ -5 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix}$

$$AB = \begin{pmatrix} 3 & 0 & 2 \\ 8 & -1 & 4 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 4 & 5 \\ -5 & 1 & 2 \\ 3 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 12 & 12 & 17 \\ 33 & 31 & 42 \\ 5 & 9 & 14 \end{pmatrix}$$

& $tr(AB) = 12+31+14 = 57$

3.) know $A^t = -A$, then $(A^k)^t = (\underbrace{A \cdots A}_{k \text{ times}})^t = \underbrace{A^t A^t \cdots A^t}_{k \text{ times}} = (A^t)^k$

so $(A^k)^t = (A^t)^k = (-A)^k = (-1)^k A^k = \begin{cases} A^k & \text{if } k \text{ even} \\ -A^k & \text{if } k \text{ odd} \end{cases}$

so A^k is sym if k even & skew sym if k odd

4.) consider $(AA^t)^t = (A^t)^t A^t = AA^t$ so AA^t sym.

now $(A^tA)^t = A^t(A^t)^t = A^tA$ so A^tA sym.

5.) consider $(AB)(B^{-1}A^{-1}) = AA^{-1} = I_n$ also consider $(B^{-1}A^{-1})(AB) = B^{-1}B = I_n$

this shows AB is invertible so by def of inverse $(AB) = B^{-1}A^{-1}$

6.) ~~know~~ $AA^{-1} = I_n$ so consider $(AA^{-1})^t = I_n^t \Rightarrow (A^{-1})^t A^t = I_n$

also $A^{-1}A = I_n$ so consider $(A^{-1}A)^t = I_n^t \Rightarrow A^t(A^{-1})^t = I_n$

$\Rightarrow$ A^t is invertible and by def of inverse $(A^t)^{-1} = (A^{-1})^t$

7.) consider

$$[A \mid I_3] = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 - R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & -1 & 1 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 - 3R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 2 & 0 & 3 & 1 & -3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & \frac{3}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

so $A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ \frac{3}{2} & \frac{1}{2} & -\frac{3}{2} \\ -1 & 0 & 1 \end{pmatrix}$

8.) consider $[A \mid I_3] = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$

$\xrightarrow{\frac{1}{2}R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_1 - 3R_2} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$

$\xrightarrow{R_1 - R_3} \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 1 & 0 & 0 & -1 & \frac{3}{2} & \frac{5}{2} \end{array}\right)$

$\xrightarrow[R_2 \leftrightarrow R_3]{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 0 & 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} \end{array}\right)$ so $A^{-1} = \begin{pmatrix} -1 & \frac{3}{2} & \frac{5}{2} \\ 1 & -\frac{3}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$

- - - - - - - - - - - - - -

9.) let $A^t A = C$ w/ $C = [c_{ij}]$ then $A = (a_{ij})$ & $A^t = [a_{ij}^t]$

so $c_{ij} = \sum_{k=1}^{n} a_{ik}^t a_{kj} = \sum_{k=1}^{n} a_{ki} a_{kj}$ so $tr(C) = \sum_{i=1}^{n} c_{ii}$

so $tr(A^t A) = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki} a_{ki} = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki}^2$ $a_{ki}^2 \geq 0$ by default

so $\sum_{i=1}^{n} \sum_{k=1}^{n} a_{ki}^2 \geq 0 \Rightarrow tr(A^t A) \geq 0.$