Ans key

1.) $\begin{cases} y' = \sqrt{y-5} \\ y(a) = b \end{cases}$

set $f(x,y) = \sqrt{y-5}$, then $f(x,y)$ is cont. for all x and $y \geq 5$

then $\frac{\partial f}{\partial y} = \frac{1}{2\sqrt{y-5}}$, then $\frac{\partial f}{\partial y}$ is cont. for all x and $y > 5$

then if $b > 5$ have unique sol'n, if $b < 5$ there is no sol'n

and if $b = 5$ there are infinitely many sol'ns

2.) $\begin{cases} y' = \ln(y-3) \\ y(a) = b \end{cases}$

set $f(x,y) = \ln(y-3)$ then $f(x,y)$ is cont. for all x and $y > 3$

then $\frac{\partial f}{\partial y} = \frac{1}{y-3}$ then $\frac{\partial f}{\partial y}$ is cont. for all x and $y \neq 3$

if $b > 3$ have unique sol'n, if $b \leq 3$ have no sol'n no case

for when there are infinitely many sol'ns

3.) $x^2 y' = 1 - x^2 + y^2 - x^2 y^2$ (separable)

$x^2 y' = 1 - x^2 + y^2(1-x^2)$ so $x^2 y' = (1-x^2)(1+y^2)$ $\Rightarrow$ $\frac{1}{1+y^2}\frac{dy}{dx} = x^{-2} - 1$

$\Rightarrow \int \frac{1}{1+y^2}\,dy = \int (x^{-2}-1)\,dx$ $\Rightarrow$ $\tan^{-1} y = \frac{-1}{x} - x + C$ so $y = \tan\left(C - x - \frac{1}{x}\right)$

4.) $\frac{dy}{dx} = 3\sqrt{xy}$ (separable)

so $\frac{1}{\sqrt{y}}\frac{dy}{dx} = 3\sqrt{x}$ $\Rightarrow$ $\int y^{-\frac{1}{2}}\,dy = \int 3x^{\frac{1}{2}}\,dx$ so $2\sqrt{y} = 2x^{\frac{3}{2}} + C$

so $y = \left(x^{\frac{3}{2}} + C\right)^2$

5.) $2xyy' = x^2 + 2y^2$

so $y' = \frac{1}{2}\frac{x}{y} + \frac{y}{x}$ sub $u = \frac{y}{x} \Rightarrow y = ux$ so $y' = u'x + u$

plugging in gives $u + xu' = \frac{1}{2}u^{-1} + u \Rightarrow xu' = \frac{1}{2}u^{-1}$ (separable)

$\Rightarrow uu' = \frac{1}{2}x$ so $\frac{1}{2}u^2 = \frac{1}{4}x^2 + C \Rightarrow u^2 = \frac{1}{2}x^2 + C$

so $\frac{y^2}{x^2} = \frac{1}{2}x^2 + C \Rightarrow y^2 = \frac{1}{2}x^4 + Cx^2$

6.) $\frac{xy'}{y} = 4x^2 + \ln y$

sub $u = \ln y$ so $u' = \frac{y'}{y}$ plugging in gives $xu' = 4x^2 + u$ (linear)

$\Rightarrow u' - \frac{1}{x}u = 4x$ so $\mu = \exp\left(\int -\frac{1}{x}dx\right) = e^{-\ln x} = x^{-1}$

$\Rightarrow (x^{-1}u)' = 4 \Rightarrow x^{-1}u = 4x + C \Rightarrow u = 4x^2 + Cx$

$\Rightarrow \ln y = 4x^2 + Cx$ so $y = e^{4x^2 + Cx}$

7.) $y_1 = x^2$, $y_2 = x^{-3}$

then $W(y_1, y_2) = \begin{vmatrix} x^2 & x^{-3} \\ 2x & -3x^{-4} \end{vmatrix} = -3x^{-2} - 2x^{-2} = -5x^{-2} \neq 0$ for all $x \neq 0$

so y_1, y_2 are linear independent everywhere except $x = 0$

then $y_1' = 2x$, $y_2' = -3x^{-4}$, $y_1'' = 2$, $y_2'' = 12x^{-5}$

so $x^2y_1'' + 2xy_1' - 6y_1 = x^2(2) + 2x(2x) - 6x^2 = 2x^2 + 4x^2 - 6x^2 = 0$ ✓

$x^2y_2'' + 2xy_2' - 6y_2 = x^2(12x^{-5}) + 2x(-3x^{-4}) - 6x^{-3} = 12x^{-3} - 6x^{-3} - 6x^{-3} = 0$ ✓

8.

8.) $y_1 = x$, $\quad y_2 = x\ln x$

then $W(y_1, y_2) = \begin{vmatrix} x & x\ln x \\ 1 & \ln x + 1 \end{vmatrix} = x\ln x + x - x\ln x = x \neq 0$ for $x \neq 0$

so y_1, y_2 are linear independent ~~for~~ for $x > 0$ (need $x > 0$ for y_2)

then $y_1' = 1$, $y_1'' = 0$, $\quad y_2' = \ln x + 1$, $y_2'' = \frac{1}{x}$

so $x^2 y_1'' - x y_1' + y_1 = x^2(0) - x(1) + x = -x + x = 0$ ✓

and $x^2 y_2'' - x y_2' + y_2 = x^2\left(\frac{1}{x}\right) - x(\ln x + 1) + x\ln x = x - x\ln x - x + x\ln x = 0$ ✓

9.) set $u(x) = \sin x \int_0^x f(t)\cos t\,dt - \cos x \int_0^x f(t)\sin t\,dt$

then $u' = \cos x \int_0^x f(t)\cos t\,dt + \sin x\, f(x)\cos x + \sin x \int_0^x f(t)\sin t\,dt - \cos x\, f(x)\sin x$

$= \cos x \int_0^x f(t)\cos t\,dt + \sin x \int_0^x f(t)\sin t\,dt$

so $u'' = f(x)\cos^2 x - \sin x \int_0^x f(t)\cos t\,dt + f(x)\sin^2 x + \cos x \int_0^x f(t)\sin t\,dt$

$= f(x) - \sin x \int_0^x f(t)\cos t\,dt + \cos x \int_0^x f(t)\sin t\,dt$

so $u'' + u = f(x) - \sin x \int_0^x f(t)\cos t\,dt + \cos x \int_0^x f(t)\sin t\,dt - \cos x \int_0^x f(t)\sin t\,dt = f(x)$

$+ \sin x \int_0^x f(t)\cos t\,dt$