Answer Key

1.) $(1+x)\,y' + y = \cos x$

$$y' + \frac{1}{1+x}y = \frac{\cos x}{1+x} \qquad \mu = \exp\left(\int \frac{1}{1+x}\,dx\right) = e^{\ln(1+x)} = 1+x$$

$$\left[(x+1)y\right]' = \cos x \qquad (x+1)\,y = \sin x + c$$

$$y = \frac{\sin x}{1+x} + \frac{c}{1+x}$$

2.) $x y' = 2y + x^3 \cos x$

$$y' - \frac{2}{x}y = x^2\cos x \qquad \mu = \exp\left(\int -\frac{2}{x}\,dx\right) = e^{-2\ln x} = x^{-2}$$

$$(x^{-2}y)' = \cos x \qquad x^{-2}y = \sin x + c \qquad y = x^2\sin x + c x^2$$

3.) $y'' - 4y = \sinh(2x)$ homog prob $y'' - 4y = 0 \Rightarrow r^2 - 4 = 0 \quad r = \pm 2$

$$y_h = c_1 e^{2x} + c_2 e^{-2x} \quad \text{w/} \quad y_1 = e^{2x},\ y_2 = e^{-2x} \qquad y_1' = 2e^{2x},\ y_2' = -2e^{-2x}$$

$$W(y_1, y_2) = \begin{vmatrix} e^{2x} & e^{-2x} \\ 2e^{2x} & -2e^{-2x} \end{vmatrix} = -2 - 2 = -4 \quad \text{have} \quad y_p = u_1 y_1 + u_2 y_2$$

$$\text{so} \begin{cases} e^{2x}u_1' + e^{-2x}u_2' = 0 \\ 2e^{2x}u_1' - 2e^{-2x}u_2' = \sinh(2x) \end{cases} \Rightarrow u_1' = \frac{1}{W(y_1,y_2)}\begin{vmatrix} 0 & e^{-2x} \\ \sinh(2x) & -2e^{-2x} \end{vmatrix} = \frac{-1}{4}\left(-e^{-2x}\sinh(2x)\right)$$

$$\Rightarrow u_1' = \frac{1}{4}e^{-2x}\sinh(2x) = \frac{1}{4}e^{-2x}\left(\frac{e^{2x} - e^{-2x}}{2}\right) = \frac{1}{8}\left(1 - e^{-4x}\right)$$

$$u_1 = \frac{1}{8}\int (1 - e^{-4x})\,dx = \frac{1}{8}\left(x + \frac{1}{4}e^{-4x}\right), \quad u_2' = \frac{-1}{4}\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & \sinh(2x) \end{vmatrix} = -\frac{1}{4}e^{2x}\sinh(2x)$$

$$u_2' = -\frac{1}{4}e^{2x}\left(\frac{e^{2x} - e^{-2x}}{2}\right) = -\frac{1}{8}\left(e^{4x} - 1\right) \qquad u_2 = -\frac{1}{8}\int (e^{4x} - 1)\,dx = -\frac{1}{8}\left(\frac{1}{4}e^{4x} - x\right)$$

$$y_p = \frac{1}{8}xe^{2x} + \frac{1}{32}e^{-2x} - \frac{1}{32}e^{2x} + \frac{1}{8}xe^{-2x}$$

$$\text{So} \quad y = c_1 e^{2x} + c_2 e^{-2x} + \frac{1}{8}xe^{2x} + \frac{1}{8}xe^{-2x} + \frac{1}{32}\left(e^{-2x} - e^{2x}\right)$$

4.) $y'' + y = \tan x$ homog. prob: $y'' + y = 0 \Rightarrow r^2 + 1 = 0$ $r = \pm i$

$y_h = c_1 \cos x + c_2 \sin x$, then $y_1 = \cos x$, $y_2 = \sin x$, $y_1' = -\sin x$, $y_2' = \cos x$

$w(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$ have $y_p = u_1 y_1 + u_2 y_2$

get $\begin{cases} (\cos x) u_1' + (\sin x) u_2' = 0 \\ (\sin x) u_1' + (\cos x) u_2' = \tan x \end{cases} \Rightarrow u_1' = \frac{1}{1} \begin{vmatrix} 0 & \sin x \\ \tan x & \cos x \end{vmatrix} = \frac{-\sin^2 x}{\cos x}$

so $u_1 = -\int \frac{\sin^2 x}{\cos x} dx = -\int \frac{1 - \cos^2 x}{\cos x} dx = -\int (\sec x - \cos x) dx = -\ln|\sec x + \tan x| + \sin x$

$u_2' = \frac{1}{1} \begin{vmatrix} \cos x & 0 \\ -\sin x & \tan x \end{vmatrix} = \sin x$ so $u_2 = \int \sin x \, dx = -\cos x$

$y_p = -\cos x \ln|\sec x + \tan x| + \sin x \cos x - \sin x \cos x = -\cos x \ln|\sec x + \tan x|$

then $y = c_1 \cos x + c_2 \sin x - \cos x \ln|\sec x + \tan x|$

5.) $x' = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} x$ $\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 + 9 = 16 - 8\lambda + \lambda^2 + 9$

$= \lambda^2 - 8\lambda + 25$

$\lambda^2 - 8\lambda + 25 = 0 \Rightarrow \lambda = \frac{8 \pm \sqrt{64 - 100}}{2} = \frac{8 \pm \sqrt{-36}}{2} = \frac{8 \pm 6i}{2} = 4 \pm 3i$

$\lambda = 4 + 3i$

$(A - (4 + 3i)) \vec{v} = 0 \Rightarrow \left(\begin{array}{cc|c} -3i & -3 & 0 \\ 3 & -3i & 0 \end{array} \right) \rightarrow \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array} \right)$ $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$

$x(t) = e^{4t} e^{3it} \begin{pmatrix} i \\ 1 \end{pmatrix} = e^{4t} \left((\cos(3t) + i \sin(3t)) \begin{pmatrix} i \\ 1 \end{pmatrix} \right) = e^{4t} \begin{pmatrix} i \cos(3t) - \sin(3t) \\ \cos(3t) + i \sin(3t) \end{pmatrix}$

$= e^{4t} \left[\begin{pmatrix} -\sin(3t) \\ \cos(3t) \end{pmatrix} + i \begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix} \right]$ so $x(t) = c_1 e^{4t} \begin{pmatrix} -\sin(3t) \\ \cos(3t) \end{pmatrix} + c_2 e^{4t} \begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix}$

6.) $x' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} x$ $\qquad \det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) - 6$

$= -4 + \lambda - 4\lambda + \lambda^2 - 6$

$= \lambda^2 - 3\lambda - 10$

have $\lambda^2 - 3\lambda - 10 = 0 \Rightarrow (\lambda - 5)(\lambda + 2) = 0 \qquad \lambda = 5, -2$

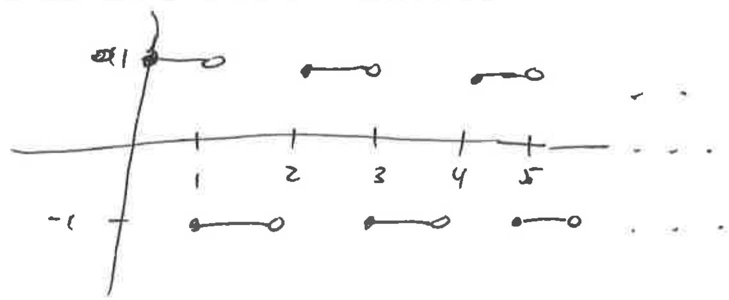
$\underline{\lambda = 5}$: $(A - 5I)\vec{v} = 0 \Rightarrow \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 3 & -6 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\underline{\lambda = -2}$: $(A + 2I)\vec{v} = 0 \Rightarrow \left(\begin{array}{cc|c} 6 & 2 & 0 \\ 3 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 3 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$

then $x(t) = c_1 e^{5t}\begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-2t}\begin{pmatrix} 1 \\ -3 \end{pmatrix}$

7.) $f(t) = (-1)^{[[t+3]]}$

graph



notice $f'(t) = 0$

$t_n = n \quad n \geq 1$ and $j_f(t_n) = 2$

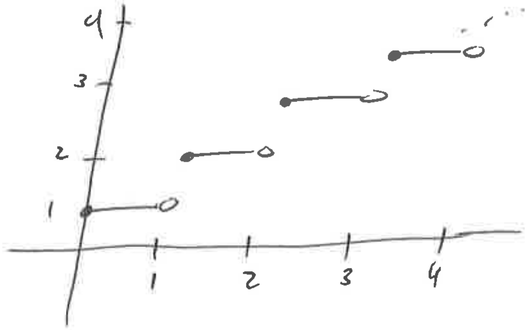
and $f(0) = 1$

so $\mathcal{L}\{f'(t)\} = sF(s) - f(0) - \sum_{n=1}^{\infty} j_f(t_n) e^{-st_n} \Rightarrow 0 = sF(s) - 1 - \sum_{n=1}^{\infty} 2e^{-sn}$

$\Rightarrow sF(s) = 1 + 2\sum_{n=1}^{\infty} (e^{-s})^n = 1 + 2\left(-1 + \sum_{n=0}^{\infty} (e^{-s})^n\right) = 1 + 2\left(-1 + \frac{1}{1-e^{-s}}\right)$

$sF(s) = 1 - 2 + \frac{2}{1-e^{-s}} \Rightarrow F(s) = -\frac{1}{s} + \frac{2}{s(1-e^{-s})}$

8.) $f(t) = 1 + [[t]]$ graph



notice $f'(t) = 0$

$t_n = n \quad n \geq 1, \quad j_f(t_n) = 1$

and $f(0) = 1$

so $\mathcal{L}(f'(t)) = sF(s) - f(0) - \sum_{n=1}^{\infty} j_f(t_n) e^{-st_n} \Rightarrow 0 = sF(s) - 1 - \sum_{n=1}^{\infty} e^{-sn}$

$\Rightarrow sF(s) = \sum_{n=0}^{\infty} (e^{-s})^n \Rightarrow F(s) = \frac{1}{s} \frac{1}{1-e^{-s}}$

9.) $\begin{cases} x'' + 4x = \cos t \\ x(0) = x'(0) = 0 \end{cases}$ $\quad \mathcal{L}(x'' + 4x) = \mathcal{L}(\cos t)$

$\Rightarrow s^2 X(s) - s\,x(0) - x'(0) + 4X(s) = \frac{s}{s^2+1}$

$\Rightarrow (s^2+4)X(s) = \frac{s}{s^2+1} \Rightarrow X(s) = \frac{s}{(s^2+4)(s^2+1)} = \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1}$

$\Rightarrow s = (As+B)(s^2+1) + (Cs+D)(s^2+4) \Rightarrow s = (A+C)s^3 + (B+D)s^2 + (A+4C)s + B+4D$

$\Rightarrow \begin{cases} A+C=0 \\ B+D=0 \\ A+4C=1 \\ B+4D=0 \end{cases} \Rightarrow D = B = 0, \quad A = -C \Rightarrow 3C = 1 \quad C = \frac{1}{3} \quad A = -\frac{1}{3}$

so $X(s) = -\frac{1}{3} \frac{s}{s^2+4} + \frac{1}{3} \frac{s}{s^2+1} \Rightarrow x(t) = -\frac{1}{3}\cos(2t) + \frac{1}{3}\cos t$

you could use the convolution but b/c of trig identities it may look different than this.

10.) $\begin{cases} x'' + 4x' + 8x = e^{-t} \\ x(0) = x'(0) = 0 \end{cases}$

$\mathcal{L}(x'') + \mathcal{L}(4x') + \mathcal{L}(8x) = \frac{1}{s+1}$

$s^2 X(s) - sx(0) - x'(0) + 4(sX(s) - x(0)) + 8X(s) = \frac{1}{s+1}$

$\Rightarrow (s^2 + 4s + 8) X(s) = \frac{1}{s+1} \Rightarrow X(s) = \frac{1}{(s+1)(s^2+4s+8)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4s+8}$

$\Rightarrow 1 = A(s^2+4s+8) + (Bs+C)(s+1) = (A+B)s^2 + (4A+B+C)s + 8A + C$

$\Rightarrow \begin{cases} A+B=0 \\ 4A+B+C=0 \\ 8A+C=1 \end{cases} \Rightarrow$ $B = -A$, $C = 1 - 8A$ $\Rightarrow$ $4A - A + 1 - 8A = 0$

$-5A = -1 \Rightarrow A = \frac{1}{5}, \quad B = \frac{-1}{5}, \quad C = \frac{-3}{5}$

so $X(s) = \frac{1}{5}\left(\frac{1}{s+1}\right) - \frac{1}{5}\left(\frac{s+3}{s^2+4s+8}\right) = \frac{1}{5}\left(\frac{1}{s+1}\right) - \frac{1}{5}\left(\frac{s+2+1}{(s+2)^2+4}\right)$

$= \frac{1}{5}\left(\frac{1}{s+1}\right) - \frac{1}{5}\left[\frac{s+2}{(s+2)^2+4} + \frac{1}{2}\frac{2}{(s+2)^2+4}\right]$

shift of cos(2t)   shift of sin(2t)   both by −2

$x(t) = \frac{1}{5}e^{-t} - \frac{1}{5}e^{-2t}\cos(2t) - \frac{1}{10}e^{-2t}\sin(2t)$

11.) $\begin{cases} tx'' - 2x' + tx = 0 \\ x(0) = x'(0) = 0 \end{cases}$

$\mathcal{L}(tx'') - 2\mathcal{L}(x') + \mathcal{L}(tx) = 0$

$-\frac{d}{ds}\left(s^2 X(s) - sx(0) - x'(0)\right) - 2\left(sX(s) - x(0)\right) - \frac{d}{ds}X(s) = 0$

$\Rightarrow -\frac{d}{ds}(s^2 X) - X' - 2sX = 0 \Rightarrow -s^2X' - 2sX - 2sX - X' = 0$

$\Rightarrow (s^2+1)X' = -4sX \Rightarrow \frac{X'}{X} = \frac{-4s}{s^2+1} \Rightarrow \ln|X| = -2\ln(s^2+1) + C$

so $X(s) = C(s^2+1)^{-2} = \frac{C}{(s^2+1)^2} = C\left(\frac{1}{s^2+1}\right)\cdot\left(\frac{1}{s^2+1}\right)$ so $x(t) = c(\sin t) * (\sin t)$

sin t

$\Rightarrow x(t) = c\int_0^t \sin\tau \sin(t-\tau)\,d\tau$

$= \frac{c}{2}\int_0^t (\cos(2\tau - t) - \cos t)\,d\tau = \frac{c}{2}\left(\frac{1}{2}\sin(2\tau - t) - \tau\cos t\right)\Big|_0^t$

$= \frac{c}{2}\left(\frac{1}{2}\sin t - \frac{1}{2}\sin(-t) - t\cos t\right) = \frac{c}{2}\left(\frac{1}{2}\sin t + \frac{1}{2}\sin t - t\cos t\right)$

$= \frac{c}{2}\left(\sin t - t\cos t\right)$

12.) $\begin{cases} tx'' + (t-2)x' + x = 0 \\ x(0) = x'(0) = 0 \end{cases}$

$\mathcal{L}(tx'') + \mathcal{L}(tx') - 2\mathcal{L}(x') + \mathcal{L}(x) = 0$

$-\frac{d}{ds}\left(s^2X(s) - sx(0) - x'(0)\right) - \frac{d}{ds}\left(sX(s) - x(0)\right) - 2\left(sX(s) - x(0)\right) + X(s) = 0$

$\Rightarrow -\frac{d}{ds}(s^2X) - \frac{d}{ds}(sX) - 2sX + X = 0 \Rightarrow -2sX - s^2X' - sX' - X - 2sX + X = 0$

$s(s+1)X' + 4sX = 0 \Rightarrow (s+1)X' + 4X = 0 \Rightarrow \frac{X'}{X} = \frac{-4}{s+1} \Rightarrow \ln|X| = -4\ln(s+1) + C$

$\Rightarrow X(s) = \frac{c}{(s+1)^4} \leftarrow$ shift of $\frac{t^3}{3!}$ so $x(t) = ce^{-t}\frac{t^3}{3!} = \frac{c}{6}e^{-t}t^3$

13.) $\begin{cases} tx'' + x' + tx = 0 \\ x(0) = 1, \quad x'(0) = 0 \end{cases}$

$\mathcal{L}(tx'') + \mathcal{L}(x') + \mathcal{L}(tx) = 0$

$-\frac{d}{ds}\left(s^2X(s) - sx(0) - x'(0)\right) + sX(s) - x(0) - \frac{d}{ds}(X) = 0$

$\Rightarrow -\frac{d}{ds}(s^2X - s) + sX - 1 - X' = 0 \Rightarrow -s^2X' - 2sX + 1 + sX - 1 - X' = 0$

$(s^2+1)X' + sX = 0 \Rightarrow \frac{X'}{X} = \frac{s}{s^2+1} \Rightarrow \ln|X| = \frac{1}{2}\ln(s^2+1) + C \Rightarrow X(s) = \frac{c}{\sqrt{s^2+1}} = \frac{c}{s}\left(1 + \frac{1}{s^2}\right)^{-\frac{1}{2}}$

so $X(s) = \frac{c}{s}\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}\frac{1}{s^{2k}} = \frac{c}{s}\left(1 - \frac{1}{2}\frac{1}{s^2} + \frac{-\frac{1}{2}(-\frac{3}{2})}{2!}\frac{1}{s^4} + \frac{-\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}\frac{1}{s^6} + \cdots\right)$

$\uparrow t \qquad \uparrow \frac{t^3}{3!} \qquad \uparrow \frac{t^5}{5!} \quad \cdots \quad \frac{t^{2n+1}}{(2n+1)!}$

$= c\left(\frac{1}{s} - \frac{1}{2}\frac{1}{s^3} + \frac{3}{2^2}\frac{1}{2!}\frac{1}{s^5} - \frac{5\cdot3\cdot1}{2^3}\frac{1}{3!}\frac{1}{s^7} + \cdots\right)$

~~so x(t) =~~

$\uparrow 1 \qquad \uparrow \frac{t^2}{2!} \qquad \uparrow \frac{t^4}{4!} \qquad \uparrow \frac{t^6}{6!} \quad \cdots \quad \frac{t^{2n}}{(2n)!}$

so $x(t) = c\sum_{n=0}^{\infty}(-1)^n\frac{(1\cdot3\cdot5\cdot7\cdots)}{2^n(2n)!\,n!}t^{2n} = c\sum_{n=0}^{\infty}\frac{(-1)^n n!}{2^n(2n)!\,n!\,(2\cdot4\cdot6\cdots)}t^{2n} = c\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{2^n(2\cdot4\cdot6\cdots)(2n)!}$