MATH 3113 Pr. Ex 3 Soln's

1.) $A = \begin{pmatrix} 5 & 2 \\ 9 & 4 \end{pmatrix}$, $B(t) = \begin{pmatrix} t & t^3 \\ t^2 & t^7 \end{pmatrix}$

$\det(A) = \begin{vmatrix} 5 & 2 \\ 9 & 4 \end{vmatrix} = 20 - 18 = 2$ $\quad \det(B) = \begin{vmatrix} t & t^3 \\ t^2 & t^7 \end{vmatrix} = t^8 - t^5 = t^5(t^3-1)$

$B^{-1}(t) = \dfrac{1}{t^5(t^3-1)} \begin{pmatrix} t^7 & -t^3 \\ -t^2 & t \end{pmatrix}$ $\quad B^{-1}(t)$ doesn't exist at $t = 0, 1$

$\dfrac{dB}{dt} = \begin{pmatrix} 1 & 3t^2 \\ 2t & 7t^6 \end{pmatrix}$

2.) $A = \begin{pmatrix} 7 & 5 \\ 1 & -2 \end{pmatrix}$ $\quad B(t) = \begin{pmatrix} \ln t & t \\ \sin t & \cos t \end{pmatrix}$

$\det(A) = \begin{vmatrix} 7 & 5 \\ 1 & -2 \end{vmatrix} = -14 - 5 = -19$ $\quad \det(B) = \begin{vmatrix} \ln t & t \\ \sin t & \cos t \end{vmatrix} = \cos t \ln t - t \sin t$

$B^{-1}(t) = \dfrac{1}{\cos t \ln t - t \sin t} \begin{pmatrix} \cos t & -t \\ -\sin t & \ln t \end{pmatrix}$ $\quad B^{-1}(t)$ doesn't exist for $t \le 0$ as $\ln t$ not defined there and $\det B \neq 0$ on $t > 0$

$\dfrac{dB}{dt} = \begin{pmatrix} \frac{1}{t} & 1 \\ \cos t & -\sin t \end{pmatrix}$

3.) $x' = \begin{pmatrix} 4 & 2 \\ 3 & -1 \end{pmatrix} x = Ax$ for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Consider $\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) - 6 = \lambda^2 - 3\lambda - 10 = (\lambda - 5)(\lambda + 2)$

eigenvalues are $\lambda = 5, -2$ at $\lambda = 5$: $A - 5I = \begin{pmatrix} -1 & 2 \\ 3 & -6 \end{pmatrix} \to \begin{pmatrix} -1 & 2 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

at $\lambda = -2$: $A + 2I = \begin{pmatrix} 6 & 2 \\ 3 & 1 \end{pmatrix} \to \begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

so $X(t) = c_1 e^{5t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + c_2 e^{-2t} \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

4i) $x' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} x = Ax$ for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

consider $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)$

eigenvalues are $\lambda = 3, -1$ at $\lambda = 3$: $A - 3I = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

at $\lambda = -1$: $A + I = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

so $x(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

5i) $x' = \begin{pmatrix} 4 & -3 \\ 3 & 4 \end{pmatrix} x = Ax$ for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = (4-\lambda)^2 + 9 = \lambda^2 - 8\lambda + 25$ can't factor so

$\lambda = \dfrac{8 \pm \sqrt{64 - 100}}{2} = \dfrac{8 \pm \sqrt{-36}}{2} = \dfrac{8 \pm 6i}{2} = 4 \pm 3i$ complex pairs

pick $\lambda = 4 + 3i$ so $A - (4+3i)I = \begin{pmatrix} -3i & -3 \\ 3 & -3i \end{pmatrix} \to \begin{pmatrix} i & 1 \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

so for $\lambda = 4 - 3i$ get $v_2 = \bar{v}_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

then (complex) $x(t) = e^{4t}\begin{pmatrix} 1 \\ -i \end{pmatrix} e^{3it} = e^{4t}\begin{pmatrix} 1 \\ -i \end{pmatrix}(\cos(3t) + i\sin(3t)) = e^{4t}\left[\begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix} + i\begin{pmatrix} \sin(3t) \\ -\cos(3t) \end{pmatrix}\right]$

then $x(t) = c_1 e^{4t}\begin{pmatrix} \cos(3t) \\ \sin(3t) \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} \sin(3t) \\ -\cos(3t) \end{pmatrix}$

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6) $x' = \begin{pmatrix} 1 & -5 \\ 1 & -1 \end{pmatrix} x = Ax$ for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -5 \\ 1 & -1-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda) + 5 = \lambda^2 + 4 \Rightarrow \lambda = \pm 2i$

pick $\lambda = 2i \Rightarrow A - 2iI = \begin{pmatrix} 1-2i & -5 \\ 1 & -1-2i \end{pmatrix} \to \begin{pmatrix} 1 & -1-2i \\ 0 & 0 \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} 1+2i \\ 1 \end{pmatrix}$

so complex $x(t) = \begin{pmatrix} 1+2i \\ 1 \end{pmatrix} e^{2it} = \begin{pmatrix} 1+2i \\ 1 \end{pmatrix}(\cos(2t) + i\sin(2t)) = \begin{pmatrix} \cos(2t) - 2\sin(2t) \\ \cos(2t) \end{pmatrix} + i\begin{pmatrix} \sin(2t) + 2\cos(2t) \\ \sin(2t) \end{pmatrix}$

then $x(t) = c_1\begin{pmatrix} \cos(2t) - 2\sin(2t) \\ \cos(2t) \end{pmatrix} + c_2\begin{pmatrix} \sin(2t) + 2\cos(2t) \\ \sin(2t) \end{pmatrix}$

7.) $f(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ 1 & t > 1 \end{cases}$

$$\mathcal{L}(f(t)) = \int_0^\infty f(t) e^{-st}\, dt = \int_0^1 t^2 e^{-st} dt + \int_1^\infty e^{-st} dt = -\frac{1}{s} t^2 e^{-st}\Big|_0^1 + \frac{2}{s}\int_0^1 t e^{-st} dt - \frac{1}{s} e^{-st}\Big|_1^\infty$$

$u = t^2 \quad dv = e^{-st} dt$

$du = 2t\,dt \quad v = \frac{-1}{s} e^{-st}$

$$= -\frac{1}{s}e^{-s} + \frac{2}{s}\int_0^1 t e^{-st} dt - \frac{1}{s}e^{-s} = -\frac{2}{s}e^{-s} + \frac{2}{s}\left(-\frac{1}{s} t e^{-st}\Big|_0^1 + \frac{1}{s}\int_0^1 e^{-st} dt\right)$$

$u = t \quad dv = e^{-st} dt$

$du = dt \quad v = -\frac{1}{s} e^{-st}$

$$= \frac{-2}{s}e^{-s} + \frac{2}{s}\left(-\frac{1}{s}e^{-s} - \frac{1}{s^2} e^{-st}\Big|_0^1\right)$$

$$= \frac{-2}{s}e^{-s} + \frac{2}{s}\left(-\frac{1}{s}e^{-s} + \frac{1}{s^2} - \frac{1}{s^2}e^{-s}\right) = -\frac{2}{s}e^{-s} - \frac{2}{s^2}e^{-s} + \frac{2}{s^3} - \frac{2}{s^3}e^{-s}$$

8.) $f(t) = \begin{cases} e^t & 0 \le t \le 3 \\ 4 & t > 3 \end{cases}$

so $$\mathcal{L}(f(t)) = \int_0^\infty f(t) e^{-st} dt = \int_0^3 e^t e^{-st} dt + \int_3^\infty 4e^{-st} dt = \int_0^3 e^{(1-s)t} dt + 4\int_3^\infty e^{-st} dt$$

$$= \frac{1}{1-s} e^{(1-s)t}\Big|_0^3 - \frac{4}{s} e^{-st}\Big|_3^\infty = \frac{1}{1-s}\left(e^{3(1-s)} - 1\right) + \frac{4}{s}e^{-3s}$$

9.) $$\frac{d}{dt}(\sin(tA)) = \frac{d}{dt}\left(\sum_{n=0}^\infty (-1)^n \frac{t^{2n+1} A^{2n+1}}{(2n+1)!}\right) = \sum_{n=0}^\infty (-1)^n \frac{(2n+1) t^{2n} A^{2n+1}}{(2n+1)!}$$

$$= A \sum_{n=0}^\infty (-1)^n \frac{t^{2n} A^{2n}}{(2n)!}\left(\frac{2n+1}{2n+1}\right) = A \sum_{n=0}^\infty (-1)^n \frac{t^{2n} A^{2n}}{(2n)!} = A \cos(tA)$$