Practice Exam 2 soln's

1) $y_1 = x, \quad y_2 = x\ln x, \quad y_3 = x^2 \ln x$

$$w(y_1, y_2, y_3) = \begin{vmatrix} x & x\ln x & x^2 \ln x \\ 1 & 1+\ln x & x + 2x\ln x \\ 0 & \frac{1}{x} & 3 + 2\ln x \end{vmatrix}$$

$$= x \begin{vmatrix} 1+\ln x & x+2x\ln x \\ \frac{1}{x} & 3+2\ln x \end{vmatrix} - x\ln x \begin{vmatrix} 1 & x+2x\ln x \\ 0 & 3+2\ln x \end{vmatrix} + x^2 \ln x \begin{vmatrix} 1 & 1+\ln x \\ 0 & \frac{1}{x} \end{vmatrix}$$

$$= x\left[(1+\ln x)(3+2\ln x) - (1+2\ln x)\right] - x\ln x(3+2\ln x) + x\ln x$$

$$= x(1+\ln x)(3+2\ln x) - x(1+2\ln x) - x\ln x(3+2\ln x) + x\ln x$$

$$= x(3+2\ln x) - x(1+\ln x) = 2x + x\ln x = x(2+\ln x) = 0 \text{ only at } x=0$$

but $\ln x$ domain is $x>0$ so lin. indep. on $x>0$

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2) $y_1 = x, \quad y_2 = \cos(\ln x), \quad y_3 = \sin(\ln x)$

$$w(y_1, y_2, y_3) = \begin{vmatrix} x & \cos(\ln x) & \sin(\ln x) \\ 1 & \frac{-\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \\ 0 & \frac{\sin(\ln x) - \cos(\ln x)}{x^2} & \frac{-\sin(\ln x) - \cos(\ln x)}{x^2} \end{vmatrix}$$

$$= x \begin{vmatrix} \frac{-\sin(\ln x)}{x} & \frac{\cos(\ln x)}{x} \\ \frac{\sin(\ln x) - \cos(\ln x)}{x^2} & \frac{-\sin(\ln x) - \cos(\ln x)}{x^2} \end{vmatrix} - \cos(\ln x) \begin{vmatrix} 1 & \frac{\cos(\ln x)}{x} \\ 0 & \frac{-\sin(\ln x) - \cos(\ln x)}{x^2} \end{vmatrix} + \sin(\ln x) \begin{vmatrix} 1 & \frac{-\sin(\ln x)}{x} \\ 0 & \frac{\sin(\ln x) - \cos(\ln x)}{x^2} \end{vmatrix}$$

$$= \frac{1}{x^2}\left\{\sin^2(\ln x) + \sin(\ln x)\cos(\ln x) - \sin(\ln x)\cos(\ln x) + \cos^2(\ln x)\right\} + \frac{1}{x^2}\left[\cos(\ln x)\sin(\ln x) + \cos^2(\ln x)\right]$$

$$+ \frac{1}{x^2}\left[\sin^2(\ln x) - \sin(\ln x)\cos(\ln x)\right]$$

$$= \frac{1}{x^2} + \frac{1}{x^2} = \frac{2}{x^2} \neq 0 \text{ on all } x$$ so lin. indep for all x

3.) $2y'' - 7y' + 3y = 0$

the char. eq. is $2r^2 - 7r + 3 = 0$ $\Rightarrow$ $(2r-1)(r-3) = 0$ so $r = \frac{1}{2}, 3$

$\Rightarrow y_h = c_1 e^{\frac{1}{2}x} + c_2 e^{3x}$

4.) $9y'' + 6y' + 4y = 0$

the char. eq. is $9r^2 + 6r + 4 = 0$ so $r = \dfrac{-6 \pm \sqrt{36 - 144}}{18} = \dfrac{-6 \pm \sqrt{-108}}{18}$

$\Rightarrow r = \dfrac{-6 \pm 2i\sqrt{27}}{18} = \dfrac{-6 \pm 6i\sqrt{3}}{18} = \dfrac{-1 \pm i\sqrt{3}}{3}$

so $y_h = e^{-\frac{1}{3}x}\left(c_1 \cos\left(\frac{\sqrt{3}}{3}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{3}x\right)\right)$

5.) $y'' + 9y = 2x^2$

$y_h = c_1 \cos(3x) + c_2 \sin(3x)$ guess $y_p = Ax^2 + Bx + C$

then $y_p' = 2Ax + B$ and $y_p'' = 2A$ plugging in gives

$2A + 9Ax^2 + 9Bx + 9C = 2x^2$ $\Rightarrow$ $9Ax^2 + 9Bx + 2A + 9C = 2x^2$

$\Rightarrow \begin{cases} 9A = 2 \\ 9B = 0 \\ 2A + 9C = 0 \end{cases}$ $\Rightarrow$ $A = \frac{2}{9}$, $B = 0$ then $\frac{4}{9} + 9C = 0$ $\Rightarrow$ $9C = -\frac{4}{9}$ or $C = \frac{-4}{81}$

so $y_p = \frac{2}{9}x^2 - \frac{4}{81}$ and $y = c_1 \cos(3x) + c_2 \sin(3x) + \frac{2}{9}x^2 - \frac{4}{81}$

6) $y'' + 3y' + 2y = e^x$

$y_h = c_1 e^{-x} + c_2 e^{-2x}$ guess $y_p = Ae^x$ then $y_p' = Ae^x$, $y_p'' = Ae^x$

so $Ae^x + 3Ae^x + 2Ae^x = e^x \Rightarrow 6A = 1$ or $A = \frac{1}{6}$

so $y_p = \frac{1}{6}e^x$ and $y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{6}e^x$

7.) $y'' + 9y = 2\sec(3x)$

$y_h = c_1\cos(3x) + c_2\sin(3x)$ try $y_p = u_1 y_1 + u_2 y_2$ set $y_1 = \cos(3x)$, $y_2 = \sin(3x)$

then $y_1' = -3\sin(3x)$, $y_2' = 3\cos(3x)$ and $W(y_1, y_2) = \begin{vmatrix} \cos(3x) & \sin(3x) \\ -3\sin(3x) & 3\cos(3x) \end{vmatrix} = 3$

so have $\cos(3x)u_1' + \sin(3x)u_2' = 0$

$-3\sin(3x)u_1' + 3\cos(3x)u_2' = 2\sec(3x)$

so $u_1' = \frac{1}{W(y_1,y_2)}\begin{vmatrix} 0 & \sin(3x) \\ 2\sec(3x) & 3\cos(3x) \end{vmatrix} = \frac{1}{3}\left(-2\sec(3x)\sin(3x)\right) = \frac{-2}{3}\frac{\sin(3x)}{\cos(3x)}$

so $u_1 = \frac{-2}{3}\int \frac{\sin(3x)}{\cos(3x)}dx = \frac{2}{9}\ln|\cos(3x)|$

$u_2' = \frac{1}{W(y_1,y_2)}\begin{vmatrix} \cos(3x) & 0 \\ -3\sin(3x) & 2\sec(3x) \end{vmatrix} = \frac{1}{3}(2) = \frac{2}{3}$ so $u_2 = \int \frac{2}{3}dx = \frac{2}{3}x$

so $y_p = \frac{2}{9}\cos(3x)\ln|\cos(3x)| + \frac{2}{3}x\sin(3x)$

and $y = c_1\cos(3x) + c_2\sin(3x) + \frac{2}{9}\cos(3x)\ln|\cos(3x)| + \frac{2}{3}x\sin(3x)$

8.) $y'' + y = \csc^2 x$

$y_h = c_1\cos x + c_2 \sin x$ set $y_p = u_1 y_1 + u_2 y_2$ and $y_1 = \cos x$ $y_2 = \sin x$

$y_1' = -\sin x$ $y_2' = \cos x$

and $W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1$ $\begin{cases} \cos x\, u_1' + \sin x\, u_2' = 0 \\ -\sin x\, u_1' + \cos x\, u_2' = \csc^2 x \end{cases}$

$u_1' = \frac{1}{1}\begin{vmatrix} 0 & \sin x \\ \csc^2 x & \cos x \end{vmatrix} = -\sin x \csc^2 x = -\csc x$

so $u_1 = \int -\csc x\, dx = -\ln|\csc x + \cot x|$

$u_2' = \frac{1}{1}\begin{vmatrix} \cos x & 0 \\ -\sin x & \csc^2 x \end{vmatrix} = \frac{\cos x}{\sin^2 x}$ $u_2 = \int \frac{\cos x}{\sin^2 x}\,dx = -\frac{1}{\sin x} = -\csc x$

$y_p = -\cos x \ln|\csc x + \cot x| + -\csc x \sin x = -1 - \cos x \ln|\csc x + \cot x|$

$y = c_1 \cos x + c_2 \sin x - 1 - \cos x \ln|\csc x + \cot x|$

9.) $ay'' + by' + cy = 0$ (*)

So $W = y_1 y_2' - y_1' y_2$ so $\frac{dw}{dx} = y_1' y_2' + y_1 y_2'' - y_1' y_2' - y_1'' y_2$

$= y_1 y_2'' - y_2 y_1''$

then $a\frac{dw}{dx} = a(y_1 y_2'' - y_2 y_1'') = y_1(a y_2'') - y_2(a y_1'') = y_1(-by_2' - cy_2) - y_2(-by_1' - cy_1)$

$= -b y_1 y_2' + b y_1' y_2 = -b(y_1 y_2' - y_1' y_2) = -bw$

hence $a\frac{dw}{dx} = -bw$ Yes it can be solved it's a separable eq.

and $w(x) = A\exp\left(-\int \frac{b}{a}\,dx\right)$