Practice Exam 3 Solutions

1.) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{n^2+1}{2n^2+3}\right)^n$ use ~~ratio~~ root test, then $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{n^2+1}{2n^2+3} = \frac{1}{2}$

so $\frac{1}{2} < 1$ then by root test the series converges absolutely

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2.) $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{n!}$, use ratio test

1st $\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \frac{(n+1)^n \cdot (n+1)}{(n+1) \cdot n^n} = \left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n$

then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e > 1$ so by ratio test, the series diverges

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3.) $f(x) = \sin x$, $a = \frac{\pi}{2}$

$f'(x) = \cos x$ $\qquad f^{(0)}(\frac{\pi}{2}) = 1$

$f''(x) = -\sin x$ $\qquad f'(\frac{\pi}{2}) = 0$

$f'''(x) = -\cos x$ $\qquad f''(\frac{\pi}{2}) = -1$

$f^{(iv)}(x) = \sin x$ $\qquad f'''(\frac{\pi}{2}) = 0$

$\qquad f^{(iv)}(\frac{\pi}{2}) = 1$

so $\sum_{n=0}^{\infty} \frac{f^{(n)}(\frac{\pi}{2})}{n!}(x-\frac{\pi}{2})^n = 1 + 0(x-\frac{\pi}{2}) + \frac{-1}{2!}(x-\frac{\pi}{2})^2 + \frac{1}{4!}(x-\frac{\pi}{2})^4 - \frac{1}{6!}(x-\frac{\pi}{2})^6 + \cdots$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}(x-\frac{\pi}{2})^{2n}$

let $a_n = \frac{(-1)^n}{(2n)!}(x-\frac{\pi}{2})^{2n}$ then $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-\frac{\pi}{2}|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x-\frac{\pi}{2}|^{2n}} = \frac{|x-\frac{\pi}{2}|^2}{(2n+2)(2n+1)}$

so $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x-\frac{\pi}{2}|^2}{(2n+2)(2n+1)} = 0$ so by ratio test series converges abs.

for all x so $R = \infty$ and the interval is $(-\infty, \infty)$

4.) $f(x) = \ln x \quad , \quad a = 2$

$f'(x) = \frac{1}{x}$

$f''(x) = \frac{-1}{x^2}$

$f'''(x) = \frac{2}{x^3}$

$f^{(4)}(x) = \frac{-2 \cdot 3}{x^4}$

$\vdots$

$f^{(n)}(x) = \frac{(-1)^{n+1}(n-1)!}{x^n}$

so $f(2) = \ln 2$

and $c_n = \frac{f^{(n)}(2)}{n!} = \frac{(-1)^{n+1}(n-1)!}{2^n \cdot n!}$ for $n \geq 1$

$= \frac{(-1)^{n+1}}{n 2^n}$

~~then~~ then $f(x) = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n 2^n} (x-2)^n$

let $a_n = \frac{(-1)^{n+1}}{n 2^n}(x-2)^n$ then $\left| \frac{a_{n+1}}{a_n} \right| = \frac{|x-2|^{n+1}}{(n+1) 2^{n+1}} \cdot \frac{n 2^n}{|x-2|^n} = \frac{|x-2|}{2} \left(\frac{n}{n+1} \right)$

$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{|x-2|}{2} \left(\frac{n}{n+1} \right) = \frac{|x-2|}{2} < 1$

need by ratio test

$\Rightarrow |x-2| < 2$ so $R = 2$ and $-2 < x-2 < 2 \Rightarrow 0 < x < 4$

at $x = 0$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n 2^n} (-2)^n = \sum_{n=1}^{\infty} \frac{-1}{n}$ diverges harmonic series

at $x = 4$: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n 2^n} 2^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by AST.

so $R = 2$ and interval is $(0, 4]$

5.) $x = t^3 - 3t \quad , \quad y = t^2 - 3$

then $\frac{dx}{dt} = 3t^2 - 3 = 3(t-1)(t+1)$ and $\frac{dy}{dt} = 2t$

vertical tangents is $\frac{dy}{dt} \neq 0$ and $\frac{dx}{dt} = 0 \Rightarrow t = 1, -1$

horizontal tangents is $\frac{dy}{dt} = 0$ and $\frac{dx}{dt} \neq 0 \Rightarrow t = 0$

6.) $x = e^{\sin t}$, $y = e^{\cos t}$

$\frac{dx}{dt} = \cos t \, e^{\sin t}$, $\frac{dy}{dt} = -\sin t \, e^{\cos t}$

vertical tangents : $\frac{dy}{dt} \neq 0$, $\frac{dx}{dt} = 0$, $e^{\sin t} \neq 0$ so $\cos t = 0$ at $t = \frac{(2n+1)\pi}{2}$ n integer

horizontal tangents : $\frac{dy}{dt} = 0$ & $\frac{dx}{dt} \neq 0$, $e^{\cos t} \neq 0$ so $-\sin t = 0$ at $t = n\pi$ n integer

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7.) $x = t\sin t$, $y = t\cos t$ $\quad 0 \leq t \leq 1$

then $x'(t) = \sin t + t\cos t$, $y'(t) = \cos t - t\sin t$

$$(x')^2 + (y')^2 = (\sin t + t\cos t)^2 + (\cos t - t\sin t)^2 = \sin^2 t + 2t\cos t\sin t + t^2\cos^2 t + \cos^2 t - 2t\cos t\sin t + t^2\sin^2 t = 1 + t^2$$

so $L = \int_0^1 \sqrt{(x')^2 + (y')^2}\, dt = \int_0^1 \sqrt{1+t^2}\, dt$ $\quad t = \tan\theta$ $\quad t = 0 \Rightarrow \theta = 0$

$dt = \sec^2\theta \, d\theta$ $\quad t = 1 \Rightarrow \theta = \frac{\pi}{4}$

$1 + t^2 = 1 + \tan^2\theta = \sec^2\theta$

$$\Rightarrow L = \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta = \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\sqrt{2} + \ln|\sqrt{2}+1| - \ln|1|\right) = \frac{1}{2}\left(\sqrt{2} + \ln|\sqrt{2}+1|\right)$$

8.) $x = 1 + 3t^2$, $y = 4 + 2t^3$ $\quad 0 \leq t \leq 1$

$x'(t) = 6t$, $y'(t) = 6t^2$ so $(x'(t))^2 + (y'(t))^2 = 36t^2 + 36t^4 = 36t^2(1+t^2)$

so $L = \int_0^1 \sqrt{36t^2(1+t^2)}\, dt = 6\int_0^1 t\sqrt{1+t^2}\, dt$ $\quad u = 1+t^2$ $\quad t = 0 \Rightarrow u = 1$

$du = 2t\,dt$ $\quad t = 1 \Rightarrow u = 2$

$$\Rightarrow L = 3\int_1^2 u^{\frac{1}{2}}\, du = 2u^{\frac{3}{2}}\Big|_1^2 = 2\left(2\sqrt{2} - 1\right)$$

9.) $\vec{a} = \langle 1, 4, -7 \rangle$, $\vec{b} = \langle 2, -1, 4 \rangle$, $\vec{c} = \langle 0, 9, 18 \rangle$

then $V = |\vec{a} \cdot (\vec{b} \times \vec{c})| = \left| \begin{matrix} 1 & 4 & -7 \\ 2 & -1 & 4 \\ 0 & 9 & 18 \end{matrix} \right| = \left| \begin{vmatrix} -1 & 4 \\ 9 & 18 \end{vmatrix} - 4 \begin{vmatrix} 2 & 4 \\ 0 & 18 \end{vmatrix} - 7 \begin{vmatrix} 2 & -1 \\ 0 & 9 \end{vmatrix} \right|$

$= | -18 - 36 \quad -144 - 126 | = |-324| = 324$

10.) $\vec{a} = \langle 0, 1, 1 \rangle$, $\vec{b} = \langle 1, 1, 0 \rangle$, $\vec{c} = \langle 1, 1, 1 \rangle$

then $V = |\vec{a} \cdot (\vec{b} \times \vec{c})| = \left| \begin{matrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{matrix} \right| = \left| 0 \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \right|$

$= | 0 - 1 | = |-1| = 1$

11.) $P_1 = (3, -1, 2)$, $P_2 = (8, 2, 4)$, $P_3 = (-1, -2, -3)$

$\vec{v}_1 = \vec{P_1P_2} = \langle 5, 3, 2 \rangle$, $\vec{v}_2 = \vec{P_1P_3} = \langle -4, -1, -5 \rangle$

then $\vec{n} = \vec{v}_1 \times \vec{v}_2 = \left| \begin{matrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 3 & 2 \\ -4 & -1 & -5 \end{matrix} \right| = \vec{i} \begin{vmatrix} 3 & 2 \\ -1 & -5 \end{vmatrix} - \vec{j} \begin{vmatrix} 5 & 2 \\ -4 & -5 \end{vmatrix} + \vec{k} \begin{vmatrix} 5 & 3 \\ -4 & -1 \end{vmatrix}$

$= \langle -13, 17, 7 \rangle$

pick $\vec{r} = \langle x, y, z \rangle$ and $\vec{r}_0 = \langle 5, 3, 2 \rangle$

then $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \Leftrightarrow \langle -13, 17, 7 \rangle \cdot \langle x-5, y-3, z-2 \rangle = 0$

$-13(x-5) + 17(y-3) + 7(z-2) = 0$

$-13x + 17y + 7z = 0$

12c) $\vec{r}(t) = \langle 1+t,\ 2-t,\ 4-3t \rangle$, $5x+2y+z=1$

if plane $\parallel$ to $5x+2y+z=1$ have same normal so $\vec{n} = \langle 5, 2, 1 \rangle$

contains $\vec{r}(t)$ pick $\vec{r}(0) = \langle 1, 2, 4 \rangle$ set $\vec{r} = \langle x, y, z \rangle$
$\vec{r}_0$

then $\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0 \Rightarrow \langle 5, 2, 1 \rangle \cdot \langle x-1, y-2, z-1 \rangle = 0$

$\Rightarrow 5(x-1) + 2(y-2) + z - 1 = 0$ or $5x + 2y + z = 10$

13.) $\displaystyle\int_0^1 x^x = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n}$

1st notice $x^x = e^{x \ln x} = \displaystyle\sum_{n=0}^{\infty} \frac{(x \ln x)^n}{n!}$

consider $\displaystyle\int_0^1 x^n (\ln x)^n \, dx = \int_0^1 \frac{x^{n+1}}{x} (\ln x)^n dx$ let $w = \ln x$, $dw = \frac{1}{x} dx$; as $x \to 0^+$ $w \to -\infty$, at $x = 1$ $w = 0$

so $\displaystyle\int = \int_{-\infty}^{0} w^n e^{(n+1)w} dw$ using the reduction formula

$$\int u^k e^{au} du = \frac{1}{a} u^k e^{au} - \frac{k}{a} \int u^{k-1} e^{au} du$$

$n+1$ times get

$\displaystyle\int = \frac{(-1)^n n!}{(n+1)^{n+1}}$ so $\displaystyle\int_0^1 x^x dx = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n n!}{(n+1)^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{n+1}}$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n}$$